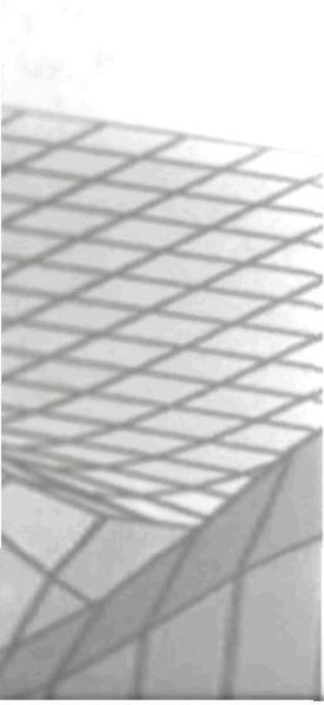
An Introduction to a Realistic Quantum Physics



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In memoriam

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Preface

It is almost thirty years that I have been teaching university courses in the field that has (almost) totally absorbed my research activity: Quantum Field Theory (QFT) and particle interactions, today known as the Standard Model (SM). I have thus had innumerable occasions to observe the uneasiness, indeed the embarassment of students when making the jump from Quantum Mechanics (QM) to QFT, the only sensible quantum description of the relativistic world, where the number of particles — quanta cannot be kept fixed. Sensible, that is a good word, but can it be really applied to QM itself? When pressed on this point the students, emerging from their institutional courses on non-relativistic QM, without exception showed how uncertain and uneasy their feeling was about a physical theory which is more than seventy years old, and permeates large sections of modern technology. As to their intellectual attitude toward QM, that is also without exception "conventionalistic" totally centered on rules and procedures, largely based on the Copenhagen interpretation and its subjectivistic probability approach.

I have always regarded (and obvioulsy I have not been alone in this) this state of affairs as very unsatisfactory, including the fact that a critical debate on such fundamental issues has remained confined to a small community of "fundamentalists", at the frontiers of physics, metaphysics and philosophy. I have thus tried to devote a (necessarily small) part of my lectures to present my point of view, which tries to reappropriate Quantum Physics (QP) to a strongly realistic view of the world, in the great tradition from Galilei to Einstein that has shaped the deep structure of modern science. But without a systematic exposition I felt that my efforts were doomed to have little or no effect at all. Thus I decided to subtract a part of my vacation time (summer 1998) to more or less futile beach talks, and

devote the following Essay to my students (and to whoever cares about the problems addressed in it) who, I hope, will benefit from being presented with ideas and analyses that are not usually found in the literature that is available to them.

This Essay is written in English for two main reasons, first this language is the common language of science, and physics students should know all too well how useful it is to read (and speak) it fluently. The second reason is that I nurture the hope that, some day it will be felt desirable to have access to its ideas in the wide world.

Milano, October 1998.

Giuliano Preparata was born in Padova, Italy on March 10, 1942. Theoretical physics was the focus of his insatiable intellectual curiosity. He succumbed to cancer in Frascati, Italy on April 24, 2000.

Contents

Chapter 1

The Fads and Fallacies of Quantum Mechanics

Quantum Physics is about one hundred years old, it represents a new image of the physical world, that sprang from the irresolvable difficulties in which classical physics (CP) sank at the end of the XIX century. And yet, and this is the peculiarity of this (unachieved) scientific revolution, after one hundred years the basic ideas and achievements of this new and profound approach to physical reality do not belong at all to the background of our culture or shape our collective understanding of and expectations about the world.

The reason for this new and strange situation, that did not occur in other scientific revolutions, like the Copernican Heliocentrism or the Einsteinian Relativity, is to my mind due to the peculiar interpretation of its basic laws and mathematical results, that since the 1930's imposed itself, and is mainly associated with the name of Niels Bohr: the Copenhagen interpretation. Such point of view suffers from a number of strange and, in the end, untenable "dogmas" about the structure of the physical world. Let's try and identify them.

First of all, according to the Copenhagen interpretation, quantum physics, or better, Quantum Mechanics (QM) addresses the microscopic world only, i.e. the world of atoms, molecules and the plethora of particles that have been the subject of intensive studies in the second part of this Century. The macroscopic world, the world of our immediate experience, is instead fully and consistently described by the venerable ideas and laws of CP. To such world there belong all physical observers, whose measuring devices have thus the sharpness and the determinism of classical physics. In this way, according to the Copenhagen interpretation, the physical world becomes intrinsically dualistic: on one side the microscopic physical systems with their chancy motions, no more describable in terms of well defined

trajectories, on the other the classical observers with their classical devices, obtaining information about the microscopic world that can only be statistical in character and, more oddly, can in no way represent the *reality* of the microscopic system, for the *random*, unpredictable interaction with the observing apparatus is an essential aspect of the physical situation, in which the observed and the observer are inextricably entangled. And the statistical distributions — the square of the wave-functions — which obey the quantum laws of evolution, do not tell the tale of the observed, but rather of the knowledge of that tale that the observer may acquire through his devices. J. A. Wheeler once put this peculiar status of affairs quite vividly when he stated that the Schrödinger wave-function bears to (the unknowable) physical reality the same relationship that a weather forecast bears to the weather. In such way the Copenhagen physicist has given up his ambition to describe the physical world *as is*, the fundamental aim of any *realist* (like Galileo Galilei and all the great classical physicists, Albert Einstein included), contenting himself to account for what *he may say* about the world, whose reality remains fundamentally inaccessible. Thus he completely embraces the position of Cardinal Bellarmino, that cost Galilei the ordeal of the process by the Inquisition and of its abject condemnation. It is interesting, but far from amusing, that with the Copenhagen QM modern science, born from the intellectual courage and achievements of a group of "realists" like Galilei, comes full circle to subscribe to the epistemological theses of "conventionalists" like Bellarmino himself.

The reason for such drastic epistemological choice, that has caused so much turmoil in the physics of this Century and is responsible for the "marginalization" of quantum physics from the main cultural trends of our time, is the realisation that particles like the electron may show, in the experiments of the Davisson–Germer type, characteristic wave-like behaviours such as interference, which intermingle two aspects of reality that appear completely irreducible, that of a particle and its discontinuous behaviour with that of a wave with its fundamentally continuous character. And it is in the typical notion of "wave–particle complementarity" of the Copenhagen "vulgata" that any hope of a realistic, objective interpretation of QM fades away, leaving in its place a well defined set of rules to "compute" the statistical distributions of the outcomes of given observations (or observables) on a statistical ensemble of identical physical systems.

As the aim of this Essay is to show the existence and describe the structure of a realistic approach to quantum physics (which does without the deeply engrained fallacies of the Copenhagen School, by denying an

independent, self consistent status of "bona-fide" physical theory to Quantum Mechanics, which only belongs to Quantum Field Theory (QFT)), it seems appropriate to give a brief historical account of the early developments of quantum physics, that in the first two decades of our Century appeared to point in a direction completely different from what came to be generally accepted through the rest of the Century.

The birth of quantum physics is usually attributed to Planck's theory of the Black Body's radiation. The application by J. Jeans of the theorem of energy equipartition to the wave-modes of a classical electromagnetic field inside a "Hohlraum", i.e. an oven, led to the well known and deprecated "ultraviolet catastrophe", according to which the energy density of the electromagnetic field inside the oven comes out to be infinite, due to the contributions of the high frequency (ν) modes, whose energy content is instead experimentally found to be highly suppressed. Planck's solution, in spectacular agreement with experiments, was really revolutionary in that it assumed that, contrary to Maxwell theory, the energy exchanges between a given mode of frequency ν and the "matter oscillators" of the oven's walls could not be continuous, whence the equipartition theorem, but could only occur in quantities, or packets or quanta of energy $\epsilon = h\nu$, where h is the famous Planck's constant ($h = 6.626\ 10^{-27}$ erg. sec.).

Such "quantization" of the energy negotiations between the electromagnetic (e.m.) field and matter was completely at variance with the continuous character of the Maxwell field and of its energy density, and dramatically called for a completely new theory of both its kinematics and dynamics. In the meantime, a few years later (1905) Albert Einstein proposed that in such energy negotiations the main actor is a new strange particle-like physical object, the "quantum" of the e.m. field, which he baptized "photon". According to Einstein's view, when energy (and momentum) get exchanged between the e.m. field and an atomic system, a given e.m. mode of frequency ν and wave vector $\vec{k}$ can only exchange energy in quantities which are integer multiples of $\epsilon = h\nu$, while the momentum must be the *same* integer multiple of $\vec{p} = \hbar\vec{k}$ ($\hbar = \frac{h}{2\pi}$). Other exchanges are strictly forbidden. In this way he could explain the oddity of the photoelectric effect, in which the ability of an electromagnetic radiation to extract electrons from a given metal surface depended not on the energy deposited on the surface but on the frequency of the radiation. The particle-like attributes of the photon were later (1921) confirmed by experiments on the diffusion of electromagnetic radiation by a charged particle, known as the Compton effect.

To a "realist", such as Einstein has been throughout his life, the "photons", which invariably appear when the e.m. field interacts with matter, must *exist* in the field, and the field must *be made of* them. Yes, but how? How can the particle properties of the photon(s) be reconciled with the wave character of the transverse Maxwell field? This was the dilemma that could, perhaps, be resolved in a fully realistic theory of the quantum world had Einstein's realism been a bit less "naive", and Planck's ideas received a bit more attention. It is not very well known, in fact, that while the great advancements of the Black Body's theory were followed by Einstein's theories of the photoelectric effect (1905) and of the specific heat of a solid (1908), a sharp debate went on between Planck and Einstein upon the physical meaning and properties of the quanta and their relationship to their quantum fields.*

According to Planck, Einstein's attitude to attribute the "photons", with their particle-like properties, a well defined physical reality beyond the acts of interaction in which they manifest themselves — discontinuous "avatars" of a wave-field — was too *naive* in the sense that in this way one could only account correctly for *one* side of reality, the interaction field–matter (observer), but totally ignored the *other* side of reality, namely the dynamical evolution of the field when nobody observes it, and no energy–momentum negotiation takes place with other (matter) fields. And the wave-like aspects of the latter, such as diffraction and interference, bear witness to the fact that the photons *cannot* be the whole story, for one side of reality cannot contradict another: the real world can't be but one! Planck's realism was thus more sophisticated than Einstein's in that it suggested the idea that the quantum field was not a well defined collection of "photons", a picture that suffices to account for the thermodynamics of the Black Body, but a *coherent* physical system defined in all space for all times (much as classical physics describes it), whose kinematics and dynamics, however, must be redefined so as to incorporate, under appropriate conditions, the appearance of photons. Of course at the time (the first decade of this Century) Max Planck did not know how to realize his ideas, but the real remarkable aspect of his realism is that it suggested in an unerring manner the way to avoid the type of nonsense that tragically convinced most of the physicists of this Century to give up the glorious tradition of realism, which led to modern science.

*For in the theory of the specific heat of a solid Einstein describes the atoms of a solid by a continuous acoustic field, whose "quantization" *à la Planck* produces a new type of quanta, the "phonons".

We may now, perhaps, begin to understand how the Copenhagen interpretation came to be perceived, at the end of the 1920's, as the only way "to make sense of the nonsensical". Soon after World War I (1923) Louis de Broglie proposed to describe in a wave-like way any (microscopic) piece of matter, thus turning upside-down Planck's (and Einstein's) approach. Now the classical physical object, the particle, gets somehow endowed with a wave-like behaviour much as the quantum object, the photon, derives such properties from the classical physical system it belongs to, the e.m. field. Whereas, however, in the quantum field approach of Planck and Einstein the field aspect is essential (and remains so after "quantization") in de Broglie's idea the field aspect is totally extrinsic, it is just an abstract mental construction grafted upon a very concrete physical object, Newton's material point. Whereas for the quantum field it was the reality of the notion of photon that had to be assessed and understood in the light of the reality of the field, in de Broglie's quantum mechanics it is the reality of the matter point that ostensibly clashes with that of its "associated wave", leading inevitably to the Copenhagen's brand of conventionalism.

But the spectacular successes in the realm of atomic physics of the newborn QM of Schrödinger and Heisenberg, that soon followed (1925) de Broglie's proposal, were judged as the best demonstration of the *usefulness* and *adequacy* of QM to account for the innumerable observations, that since a few decades challenged any understanding based on classical physics. Even though, it was clear, de Broglie waves had nothing to do with real space–time physical processes, but could only be used *to calculate* the statistical distributions of the outcomes of any observation on the particular microscopic system.

One may now see the reasons which steered physics into a new, unsavoury course that severed all ties with the great intellectual tradition of realism that marks the development of modern science. Famous is the debate between Albert Einstein and Niels Bohr† at the Solvay Conference of 1927, in which the irreducible realist (Einstein) tried to oppose with all his intellectual might the new conventionalistic (Bohr's), and basically sceptic, approach that the "stars" of QM (Bohr, Heisenberg, Schrödinger, Born, Pauli and Jordan) were then boldly shaping. To Einstein's objections, that were all based on his philosophical choices, Bohr could oppose a remarkably effective logical system that, once its odd postulates were accepted, would

†For an account by the protagonists of what happened during the Solvay Conference see Schilpp,[Schilpp (1957)].

show no gap nor weakness. And to a scientific community, whose interests in the atomistic aspects of the physical world were dramatically rising, the smooth working of Copenhagen's QM was more than enough to attribute the victory to Bohr, and push Einstein out of the quantum physics' stage, and to stop to listen to his objections, that became particularly sharp in the paper (1935) written in collaboration with Rosen and Podolsky (which exposes the so called EPR paradox).

In this way the truly successful revolution of our Century, the atomistic revolution, appears to have prevented the full development of quantum physics. Einstein, of course, was totally justified in his deep unhappiness about Bohr's extreme epistemological position, but there existed no way to challenge it in purely *logical* terms, no more than Bellarmino's arguments against Copernicus could be logically disproved by Galilei. However Galilei's realism proved then a better guide for the pursuit of scientific truth than the logically unattackable conventionalistic subtleties of the Cardinal. The question now is whether Einstein's realism could not be as successful a guide in his struggle against Bohr's triumphant conventionalism. The answer is, unfortunately, no.[‡] Surprisingly, in Einstein's quest for a realistic approach to quantum physics, he seems to have forgotten the themes of his youth and, in particular, his debate with Max Planck, all focussed on the *field* aspects of the quantum world. And instead of going back to the main field theoretical aspects of early quantum physics, and to subject to a deep analysis the relationships between fields and quanta, in search of new (non-classical) principles and laws governing physical reality, he believed that the odd statistical structure of QM was but a manifestation of the fact that QM is an *incomplete* theory of the quantum phenomena, arising from a yet to be discovered averaging over *hidden* variables, whose nature had to be researched and understood. We now know that Einstein's research programme (which, however, did not arouse much interest) was doomed to failure for, as J. Bell (1964) showed, any completion of QM through hidden variables can be experimentally tested, yielding results at variance with the predictions of QM. And the experimental evidence (Aspect, 1980) is definitely in favour of QM, in spite of the impossible difficulty of understanding the non-vanishing *particle* correlations at space-like distances, implied by the quantum mechanical description of such experiments.

It thus appears quite plausible that the keen interest that has surrounded particle physics during this Century has greatly contributed to

[‡]And this, I contend, is the reason of the utter failure of Einstein's struggle.

place the particle at the centre of theoretical investigation, endowed with a basically autonomous kinematics and dynamics, in contrast to the complete dependence of the dynamics of the quantum of early quantum physics from that of its field. And in fact the theory of the quantization of a classical field, accomplished at the end of the 1920's, came to be known as "second quantization", to distinguish it from the quantization of a system with a finite number of degrees of freedom, i.e. QM. It is again the centrality of the notion of *isolated* particle in the physical thought of our times that, to my mind, has prevented an in-depth critical analysis of the relationships between QM and the "second quantization", or Quantum Field Theory (QFT). For to the mind that embraces a wider physical realm the Copenhagen epistemology is not only hard to swallow but it is also at odds with physical observations.

Let's begin with the idea that quantum behaviour has to do with the *microscopic* world only. The well-known collective phenomena of Ferromagnetism, Superconductivity and Superfluidity, whose classical impossibility is easy to demonstrate,[§] bear witness to the fact that there exist macroscopic pieces of matter whose behaviour *cannot be described by classical physics.* If not classical, what is then the relevant physics?

Most contemporary physicists have no doubt that QFT must be involved in the still mysterious[¶] workings of these fascinating phenomena, but, beyond a few phenomenological attempts, such as the Landau-Ginsburg approach, no real headway has ever been made into this kind of physics. And, I contend, the fallacious philosophy of QM is largely responsible for this unfortunate state of affairs. Conversely, couldn't the situation improve, indeed change drastically if we were to realize the centrality of QFT and find that QM is but some kind of approximation of QFT in a well defined, limiting physical situation? This is precisely what this Essay proposes to show. But let's go on.

We have just seen that it is thus not true that, as the Copenhagen physicists claim, the quantum world is the microscopic world: there is a peculiar macroscopic system, whose behaviour can only be accounted for by quantum physics. Let's now address the other basic assumption of QM,

[§]The classical impossibility of Ferromagnetism is nicely explained in Feynman's Lectures, [Feynman *et. al* (1965)].

[¶]Naturally, this refers to the generally accepted theory of condensed matter physics, based on electrostatic interactions. For the success of a new approach to condensed matter, based on Quantum Electro Dynamics see Giuliano Preparata, QED Coherence in Matter (World Scientific, Singapore, 1995).

namely the dichotomy of the world in microscopic quantum systems and macroscopic classical observers. I wish to show the fallacy of this Copenhagen's "tenet" by simply recalling that classical physics is in fact logically inconsistent, and can thus in no way provide the theoretical description of even a part of reality, that of the observer.

The inconsistency I am referring to is "the entropy crisis", a notable example of which is the "ultraviolet catastrophe", whose resolution led to Planck's discovery of quantum physics. The fundamental works in the second half of the XIX century of J. C. Maxwell and L. Boltzmann on the statistical theory of the perfect gas exposed the fundamental connection between the statistics of the configurations in classical phase space of a large ensemble of (almost) non-interacting matter points and their thermodynamical behaviour. In particular Clausius' entropy was shown by Boltzmann to be proportional (by the proportionality constant that rightly bears his name) to the logarithm of the number of different configurations that correspond to a thermodynamical (equilibrium) state; defined by the thermodynamical variables: temperature, volume, and other possible macroscopic labels.

The wonderful achievement of directly relating the great laws of Thermodynamics to the structure and the combinatorics of the classical phase-space points — the system's configurations — hid, however, a *cadeau empoisonné* that emerged when the limit of zero (absolute) temperature came to be studied in detail. The systematic analysis that Walter Nernst [Nernst ()] conducted across the XIX and XX Centuries on the low temperature behaviour of a large number of thermodynamic systems, that finally led to his celebrated "Heat theorem" (known today as the third principle of Thermodynamics), could exclude that nature exhibits the entropy singularity that for $T \to 0$ classical statistical mechanics inevitably predicts. The reason of this singularity is quite simple: at zero (absolute) temperature there exists only one possible configuration, where all particles are at rest in their equilibrium positions, hence the entropy vanishes; but if we change the temperature by a small but finite amount, the number of classical configurations corresponding to such thermodynamical states increase by an infinite amount, thus clearly exposing an unregularizable singularity. The lack of "sufficient reason" for such zero-temperature entropy singularity excludes classical statistical mechanics, and with it classical physics altogether, from the realm of possible descriptions of the physical world. And this is evidently very bad news for the Copenhagen's view, which sees its classical world of observers lose any meaning when the temperature

becomes sufficiently low; a paradox that unfortunately is little considered and understood by the quantum mechanicians of our times.

It is amusing that at the same time (mid 1920's), when the formalism of QM was being laid out and its conventionalistic interpretation tuned up, Einstein together with the young Indian physicist S. N. Bose [Einstein (1924)] succeeded in finally showing how the quantum "perfect" gas could avoid the "entropy crisis" of the classical system, going through a phase transition, the Bose–Einstein condensation, which for $T \to 0$ brought with continuity all atoms to the unique, discrete ground state, and their entropy to vanish. And in order to accomplish this they had to revisit the early quantum theory of Planck and Einstein, where the field concept is primary and the quanta, the particles, just stem from the peculiar interactions between different fields, having thus no independent, autonomous reality, which is totally engendered in the new dynamics of the quantum field.

Had more attention been paid by the physicists' community to such developments and their deep meaning, one may legitimately dream that the story would have been quite different, and this Essay could have been written a long time ago.

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Chapter 2

Kinematics: The Descriptive Framework of Physical Reality

Κίνημα, κινήματος in the Greek language means motion. Kinematics in modern physics denotes the science of motion, i.e. the mathematical description of the correlations between the space domains spanned by the generic physical system and time. Thus central for its development is a suitable mathematical theory of space and time, which Bernhard Riemann (1854) has taught us to be deeply connected to and influenced by the physical reality one wishes to describe.

In order to keep our discussion as general as possible by space we mean a three-dimensional manifold whose points can be put in a one-to-one correspondence with the points (coordinates) $\vec{x} \in R^3$, the three-dimensional continuum of real numbers. Any such correspondence, as we know well, defines a particular observer and the principle(s) of relativity can be formulated in general by specifying the allowed class of observers and their relationships, i.e. the coordinate transformations among different observers. As for time, it is described by a one-dimensional continuum of real numbers, which depends in general on the observer, so that in the coordinate transformations from one observer to another the time coordinates can in general change: only in the Galileo–Newtonian Physics time is totally independent of the observer, i.e. there exists an "absolute" time.

A physical system can then be defined by first identifying a space-region (with respect to any allowed observer) which differs from what is perceived as empty space — the Vacuum — and where some observable properties can be identified and given numerical values. In classical physics the simplest of physical systems is the matter-point, whose space-domain reduces to a single point $\vec{x}$ and its defining physical property is the mass, a positive number depending on the chosen units. A complete set of physical observables for the matter-point is the quadruple $(\vec{x}, m)$ that *at the given time*

exhaustively describes the configuration of this physical object. However such quadruple is insufficient to describe the state of motion of the matter-point, for, in order to do this, we need to specify the correlations between the matter-point's configurations at different times t, and in particular between two infinitesimally close times. It would appear that to "label" the state of motion of the matter-point one needs to assign (at a given time) $\vec{x}$, $\vec{p} = m\dot{\vec{x}}$ (the momentum) and a number of higher time-derivatives of $\vec{x}$, but dynamics, i.e. the equations of motion, shows that all time-derivatives *of order higher than one are well defined functions of* $\vec{x}$ *and* $\vec{p}$: thus any state of motion of the matter-point can be uniquely represented by a point of the six-dimensional classical phase-space $(\vec{x}, \vec{p})$, which with the flowing of time describes a well defined trajectory $(\vec{x}(t), \vec{p}(t) = m\dot{\vec{x}}(t))$.

By "glueing" together with appropriate "internal forces" a number n of such points we may construct physical objects of increasing complexity whose phase-space dimensionality is $6n$, and the basic physical observables are $\vec{x_i}, \vec{p_i}$ $(i = 1, 2, \ldots, n)$. But the discrete systems of Newton's *Principia* do not exhaust the domain of physical phenomena, in classical physics a great deal of attention has been paid to continuous systems, the fields, whose space-domain is continuous and the number of degrees of freedom as well as of the dimensions of phase-space (PS) is infinite. Up to the great Maxwellian synthesis of electric and magnetic phenomena into electromagnetism, the field concept could be looked at as a convenient approximation of a discrete system (a collection of atoms and molecules) with a very large number (of the order of the Avogadro number, $\mathcal{N} = 6.02\ 10^{23}$) of matter-points, and this was precisely the point of view of Maxwell–Boltzmann statistical mechanics. But with the theory of electromagnetism and the utter failure of a mechanical description based on a hypothetical Aether there emerges in physics a new object, irreducible to any "mechanistic"* description, defined in *all* space for *all* times, a truly all-encompassing physical entity, which affects electrically charged matter where its intensity is perceptibly different from zero. With Einstein's General Relativity (1916) classical physics gets enriched with another genuine field, the gravitational field, which *mutatis mutandis* bears to the masses of the universe the same relation that the e.m. field bears to the electric charges. Thus, just when quantum physics (QP) was moving its first uncertain steps toward a new image of the world, classical physics (CP) had reached its zenith,

*Here, as in common parlance, "mechanistic" denotes any description that tries to reduce a given physical system to the juxtaposition of a (large) number of Newton's matter points.

delivering us a fully dychotomic world where matter is described by the relativistic generalization of the *discrete* (and discontinuous) Newton's mechanics and the fields represent that kind of continuous "alterations" of the physical Vacuum that account for the dynamical evolution of electrically charged and massive matter. This *kinematical* separation between matter (the world of "little balls") and fields (the "stressed state" of the Vacuum) and, therefore, the irreducibility of the motion of particles belonging to the former, and of the propagation of waves characterizing the latter, is perhaps the most significant feature of the picture of the world painted by classical physics, one that is deeply rooted in our immediate sensory experience.

Quantum physics has totally subverted all this, and it is thus quite remarkable that the classical distinction between matter (physical systems with a finite number of degrees of freedom) and fields has survived in the distinction between QM and the "second quantized" systems, i.e. QFT. As already anticipated, in the following I shall argue that it is just this act of hybris, which has been haunting quantum physics for almost a century, that must be finally repaired, dropping once and for all the distinction between *matter and field, and recognizing that in quantum physics there exists only* one consistent type of physical object: the quantum field.

2.1 States and observables in Classical Physics (CP)

Any physical system, that we think of at a given time t, *is*, i.e. it exists in a particular *state*. Our preliminary discussion has already indicated what a state of a classical system consists of: it is just a point of the classical phase-space (PS) whose dimension is $2f$, if f is the number of (Lagrangian) degrees of freedom that are necessary to describe its configuration. Thus the classical PS is the space of vectors of components q_i, p_i $(i = 1, 2, \ldots, f)$. For instance, for the isolated point the PS is six-dimensional, comprising the vectors $(\vec{x}, \vec{p})$, while for a rigid body it is 12-dimensional, comprising $(x_{\vec{CM}}, \phi, \theta, \psi, p_{\vec{CM}}, \vec{L})$, when $x_{\vec{CM}}$ and $p_{\vec{CM}}$ are the center of mass coordinates and momentum respectively, ϕ, θ, ψ the Euler angles and $\vec{L}$ the components of angular momentum. For a field we can also define a classical PS, which is now ∞-dimensional,comprising the values of the field and of its conjugate momentum (see later) at any space point.

As for thé physical observables, they are represented by all sufficiently well behaved functions $O(q_i, p_i)$, which at a given time t assume a well defined real numerical value, once the totality of $q_i(t)$ and $p_i(t)$ have been

determined. Thus in classical physics there is no basic distinction between the state of a system at the time t and its class of physical observables at the same time: for the state is *uniquely* determined by the values of the q_i's and p_i's, which thus constitute a complete set of observables, in the sense that any other observable can be unambiguously obtained from those values. The possibility of uniquely associating to a state of a classical system a point of the classical PS clearly stems from the fundamental hypothesis that the process of measuring the q_i's and p_i's has no perturbing effect whatsoever on the system, and the experimental indeterminacy can be made, with due care, arbitrarily small. Thus we may conclude that in CP between states and observables there is basically no distinction, for a state, a point of classical phase-space (PS), is uniquely determined by the values that the observables q_i's and p_i's attain in the state.

2.2 States and observables in Quantum Physics (QP)

The main point of departure between CP and QP is that in the quantum world there is no *unlimited observability*. As a result, states and observables are no more in a one-to-one correspondence: the process of measurement of the basic kinematical variables, the q_i's and p_i's of classical PS, loses its classical *neutrality* to become a fundamental element in the determination of the phenomenic properties of a given physical object.

Without going through the chain of logical arguments and the set of experimental facts that have led to the discovery of the mathematical structure of QP,[†] it suffices to recall that this latter postulates that the states of a quantum system belong to a complex vector space (Hilbert space) while the observables are just Hermitian operators in such space. Thus in QP a measurement process is represented by the acting on the state vector of the system of the Hermitian operator corresponding to the physical observable being measured. Only if the state vector happens to be an eigenvector of the observable $\mathbf{O}$, i.e. if[‡]

$$\mathbf{O}|o\rangle = o|o\rangle\,, \tag{2.2.1}$$

with o a real number, the measurement of $\mathbf{O}$ yields the well defined value o, belonging to its *spectrum*, i.e the set of all possible values that $\mathbf{O}$ can

[†] For this a very good book to consult still is P.A.M. Dirac, The Principles of Quantum Mechanics [Dirac (1958)].

[‡] Throughout this Essay we shall adopt Dirac's notation, which has become standard.

assume. Formally this is obtained if

- the state vectors are normalized. i.e.

$$\langle\psi|\psi\rangle = 1$$

- the value of $\mathbf{O}$, technically the "expectation value" $\langle\mathbf{O}\rangle_\psi$, in the state $|\psi\rangle$ is given by the "matrix-element"

$$\langle\mathbf{O}\rangle_\psi = \langle\psi|\mathbf{O}|\psi\rangle\,. \tag{2.2.2}$$

However if, in general, the state vector $|\psi\rangle$ is not an eingenvector of $\mathbf{O}$, the measurement of $\mathbf{O}$ will not give a sharply predictable outcome, for the application of the operator $\mathbf{O}$ to $|\psi\rangle$ discontinuously changes the state vector into an eigenstate $|o_n\rangle$ of $\mathbf{O}$, with eingenvalue o_n. It is here that our understanding (intuition) of the quantum world is put to a severe test: while we can easily understand (2.2.1), that attributes $\mathbf{O}$ a well defined value o, what does (2.2.2) really mean? We know from the general mathematical theory of linear, Hermitian operators in Hilbert spaces that $|\psi\rangle$ can be uniquely decomposed as the linear superposition:

$$|\psi\rangle = \sum_n c_n|o_n\rangle\,, \tag{2.2.3}$$

where$\{|o_n\rangle\}$ is a complete orthonormal set of eigenvectors of $\mathbf{O}$, and c_n are complex numbers. Then from (2.2.2) we obtain:

$$\langle\mathbf{O}\rangle_\psi = \sum_n |c_n|^2 o_n \quad \left(\sum_n |c_n|^2 = 1, \text{from the state normalization}\right)\,, \tag{2.2.4}$$

for which we may advance the following interpretation: $|c_n|^2$ is the probability that the measurement of $\mathbf{O}$ *reduces* the state-vector $|\psi\rangle$ to $|o_n\rangle$, yielding the outcome o_n. As a result the "expectation value" $\langle\mathbf{O}\rangle_\psi$ is just the statistical average of the outcomes of the measurements of the observable $\mathbf{O}$ upon an appropriate ensemble of identical copies of the quantum system, all in the state $|\psi\rangle$.

We may now clearly see that allowing the measuring process to perturb the state of the system has produced a profound metamorphosis in our picture of the world: the complete determinacy of the observables, the sharpness of the values they assume in a given state of CP, gives way now to quantum observations whose outcomes can only be predicted *statistically*, the state vector $|\psi\rangle$, through its coefficients c_n, being only able to tell us the probability $|c_n|^2$ of the outcome o_n. Very odd, isn't it? But the evidence

that such is the real *architecture* of the world is so overwhelming that we are well advised to get rid at once of our (classical) intuition and prejudices, and to second within ourselves the cogency and the majesty of the image of the Universe that QP has finally disclosed us.

Another dramatic consequence of the "limited observability" of QP is that classical PS loses all meaning in the quantum world. For convenience let's consider a system with only one degree of freedom, a particle moving in one dimension; its basic classical observables are its position x and its momentum p. Its generic state is a point in the two-dimensional classical PS (x,p). x and p, which we shall now denote $\mathbf{X}$ and $\mathbf{P}$ to remind us that they are not real numbers (historically referred to as c-numbers) but operators (or q-numbers, in the same terminology), are also quantum observables, whose spectra may or may not coincide with the classical values, which belong to the open real line. Such coincidence occurs if the particle motion is unconstrained, i.e. is allowed to wander over the full real line $(-\infty,+\infty)$. According to quantum kinematics the generic state can be written:

$$|\psi\rangle = \int dx\psi(x)|x\rangle \tag{2.2.5}$$

where $|x\rangle$ is a complete set of eigenvectors of $\mathbf{X}$, orthonormalised in the continuum as (δ is the Dirac δ-function) ,

$$\langle x'|x\rangle = \delta(x - x'), \tag{2.2.6}$$

which thus provide a basis for the Hilbert space of the physical states. If x is the outcome of the measurement of the observable $\mathbf{X}$, what can we say about observing $\mathbf{P}$? From the operator nature of $\mathbf{X}$ and $\mathbf{P}$ it is clear that $\mathbf{P}$ can have a well defined value on the vector $|x\rangle$ only if this is also an eingenvector of $\mathbf{P}$, and this can only happen if $\mathbf{X}$ and $\mathbf{P}$ commute, i.e. if $[\mathbf{X},\mathbf{P}] = \mathbf{XP} - \mathbf{PX} = 0$. But this can never happen, for a fundamental quantum postulate[§] demands that (i is the imaginary unit, $i^2 = -1$)

$$[\mathbf{X},\mathbf{P}] = i\hbar, \tag{2.2.7}$$

($\hbar = \frac{h}{2\pi}$, h is Planck's constant) which illuminates the pivotal role the Planck constant plays in determining the structure of the Hilbert space of states and of the canonical observables $\mathbf{X}$ and $\mathbf{P}$. Indeed, due to the

[§]The general formulation of this postulate is that in any Lagrangian system with f degrees of freedom Q_f, calling P_f the conjugate momenta, i.e. $P_f = \frac{\partial L}{\partial Q_f}$, the quantum observables associated to them obey the following commutation equations $[\mathbf{Q_f},\mathbf{Q_{f'}}] = [\mathbf{P_f},\mathbf{P_{f'}}] = 0$ and $[\mathbf{Q_f},\mathbf{P_{f'}}] = i\hbar\delta_{ff'}$.

non-commutativity of $\mathbf{X}$ and $\mathbf{P}$, the value of the latter on the eigenvector $|x\rangle$ of the former turns out to be totally unpredictable, in the sense that any outcome p is equiprobable. This is a special case of the celebrated Heisenberg principle (1926), which can be thus derived from the "canonical commutation relation" (CCR) (2.2.7). Let's define the "dispersions" of the canonical variables on the generic state $|\psi\rangle$

$$(\Delta x)^2 = (\langle\psi|\mathbf{X}^2|\psi\rangle - \langle\psi|\mathbf{X}|\psi\rangle^2) = \langle\psi|(\mathbf{X} - \bar{x})^2|\psi\rangle = \langle\psi|\mathbf{U}^2|\psi\rangle \quad (2.2.8a)$$

$$(\Delta p)^2 = (\langle\psi|\mathbf{P}^2|\psi\rangle - \langle\psi|\mathbf{P}|\psi\rangle^2) = \langle\psi|(\mathbf{P} - \bar{p})^2|\psi\rangle = \langle\psi|\mathbf{V}^2|\psi\rangle \quad (2.2.8b)$$

where by $\bar{x}$ and $\bar{p}$ we have denoted $\langle\psi|\mathbf{X}|\psi\rangle$ and $\langle\psi|\mathbf{P}|\psi\rangle$, and by $\mathbf{U}$ and $\mathbf{V}$ the operators $(\mathbf{X} - \bar{x})$ and $(\mathbf{P} - \bar{p})$ respectively. From the CCR we have

$$\langle\psi|[\mathbf{U}, \mathbf{V}]|\psi\rangle = i\hbar \quad (2.2.9)$$

which, inserting a complete basis $\sum_n |n\rangle\langle n|$ between the operators and writing $\langle n|\mathbf{U}|\psi\rangle = \alpha_n e^{i\phi_n}$ and $\langle n|\mathbf{V}|\psi\rangle = \beta_n e^{i\psi_n}$, gives

$$\sum \alpha_n\beta_n \sin(\psi_n - \phi_n) = \frac{\hbar}{2} \quad (2.2.10)$$

leading to the inequality,

$$\sum \alpha_n\beta_n \geq \sum \alpha_n\beta_n \sin(\psi_n - \phi_n) = \frac{\hbar}{2}\,. \quad (2.2.11)$$

Noting that inserting the same complete set of states in (2.2.8) yields

$$(\Delta x)^2 = \sum_n \alpha_n^2\,, \quad (2.2.12a)$$

and

$$(\Delta p)^2 = \sum_n \beta_n^2\,, \quad (2.2.12b)$$

only simple geometrical considerations (the Schwarz inequality) are needed to derive from (2.2.1) the Heisenberg inequality:

$$(\Delta x)(\Delta p) \geq \frac{\hbar}{2} \quad (2.2.13)$$

Thus if $\Delta x = 0$, as it happens when $|\psi\rangle = |x\rangle$, an eigenstate of $\mathbf{X}$, (2.2.3) requires that $\Delta p \to \infty$, i.e. the outcome of a measurement of $\mathbf{P}$ is completely uncertain. The equiprobability of any outcome can be further assessed by working out the unitary transformation from the (complete, orthonormal)

basis of the eigenstates $|x\rangle$ of $\mathbf{X}$ to the basis of the eigenstates of $\mathbf{P}$. Calling $|p\rangle$ the eigenvector of $\mathbf{P}$ with eigenvalue p, we have

$$|p\rangle = \int dx|x\rangle\langle x|p\rangle, \tag{2.2.14}$$

and sandwiching the CCR (2.2.7) between $\langle x|$ and $|p\rangle$ we easily get:

$$\int dx'\langle x|\mathbf{P}|x'\rangle\langle x'|p\rangle x' = (-i\hbar + px)\langle x|p\rangle\,, \tag{2.2.15}$$

whose immediate consequences are:

$$\langle x|\mathbf{P}|x'\rangle = -i\hbar\frac{\partial}{\partial x}\delta(x-x') \tag{2.2.16}$$

and

$$-i\hbar\frac{\partial}{\partial x}\langle x|p\rangle = p\langle x|p\rangle\,, \tag{2.2.17}$$

whose solution is

$$\langle x|p\rangle = \frac{e^{i\frac{px}{\hbar}}}{\sqrt{2\pi\hbar}} \tag{2.2.18}$$

where the normalization stems from the orthonormality of the $|x\rangle$ basis:

$$\langle x|x'\rangle = \int dp\langle x|p\rangle\langle p|x'\rangle = \delta(x-x')\,. \tag{2.2.19}$$

Thus the probability (density) of the outcome p in $|x\rangle$ is uniform, being given by

$$|\langle x|p\rangle|^2 = \frac{1}{2\pi\hbar} \tag{2.2.20}$$

in full agreement with Heisenberg's principle.

The *kinematical* discussion of QP carried in this section may seem excessively simplified and sketchy, but actually is quite complete, for its generalization to systems of any number of degrees of freedom, including quantum fields, encounters absolutely no new concept, only mathematical complications that can in no way represent a major reason of concern for the realist.

2.3 The impossibility of a trajectory is the impossibility of a realistic QM

In CP the notion of a matter-point is inseparable from that of its trajectory $\vec{x} = \vec{x}(t)$, a continuous spatial curve that the point describes as time flows. And the value of the momentum at each instant measures the "quantity" of such motion, i.e. how fast or slow the trajectory is spanned. Also, the continuity of the trajectory allows us to *identify* the particular matter-point we are actually observing and to tell it apart from any other point that is moving in the vicinity of its trajectory. In other words, its degrees of freedom are unambiguously defined, and more importantly, ***observable.*** Thus the reality of the classical mechanics of a system with a finite number of degrees of freedom (finite set of matter points) is strongly rooted both in its theoretical description — the trajectory — and in its experimental observation.

In QM the situation is drastically and dramatically different. The impossibility of attributing *simultaneously* well defined values to the coordinate **X** and the momentum **P** implies that *in principle* no trajectory can be assigned to the quantum particle. To remain in the simplest possible framework of a single degree of freedom, we can write for the generic state, as we have seen,

$$|\psi\rangle = \int dx \psi(x)|x\rangle \tag{2.3.1}$$

where

$$\psi(x) = \langle x|\psi\rangle \, ,$$

the projection of the vector $|\psi\rangle$ on the eigenstate$|x\rangle$ of **X**, is called "wave-function". According to the preceding discussion, at a given time $\psi(x)$ is *all* that QM allows us to *know* about the state in which the one-dimensional particle happens to find itself, and $|\psi(x)|^2$ is the probability (density) that a measuring apparatus located at x has to find an object that, like the classical particle, is pointlike. In particular, if $|\psi\rangle$ is the momentum eigenstate $|p\rangle$ the wave-function is given by (2.2.18), and it is thus completely delocalized, like a plane-wave of the electromagnetic field. It is here that our perception of the reality of the physical system we are interested in gets into an irresolvable crisis: on one hand the system *is* in the state $|p\rangle$, whose space structure $\langle x|p\rangle$ (see Eq. (2.2.18)) is highly suggestive of a wave-like character, on the other when we explore the space region where such system *is* we

find it endowed with the characters of a pointlike particle. There should be no doubt that both aspects pertain to the realm of physical reality, the real trouble is that they happen to be irreconcilable, i.e. *they cannot belong to the same physical object*, whose reality and identity should persist through the measuring process. And this is precisely the logical difficulty that the Copenhagen's view resolves by flatly rejecting the constraints of realism, and inventing the surprising notion of "wave–particle complementarity" (N. Bohr), by which the quantum mechanical particle behaves *sometimes* as a particle and *sometimes* as a wave, depending on the kind of measurements we perform upon it. In this way the answer to the question what *really is* the quantum particle is pushed forever outside the reach of QM, whose task is restricted to the computations of the statistical predictions of all different types of observations.

But as Einstein, Rosen and Podolsky (EPR) [Einstein *et. al* (1935)] remarked in the 1930's, the predictions of QM become even more (if possible) puzzling to even a mild realist, when the QM of a two-particle system is analyzed. The wave-function for such system is now $\psi(x_1, x_2)$, a general function of the eigenvalues of the observables $\mathbf{X_{1,2}}$, the "position operators" of the two particles. EPR showed that the general "entanglement" of $\psi(x_1, x_2)$, i.e. the fact that in general $\psi(x_1, x_2) \neq \phi(x_1)\chi(x_2)$, leads to the prediction of non vanishing particle–particle correlations for space-like separations, thus violating the principle of causality. As mentioned, careful experiments carried out in the 1980's have fully confirmed such quantum predictions, exposing in a definite way the impossibility to reconcile QM with causality as well.

The discussion of this section has, I believe, convinced us that the most severe difficulties of QM do not lie at all in the general ideas of quantum kinematics, in the fundamental distinction between the Hilbert space of states and the operator-nature of physical observables, which stems from the new "creative" nature of the observations or, more generally, of the interactions among different physical systems. Where such difficulties become insurmountable is in the quantum-mechanical assumption that through the general ideas and postulates of quantum kinematics one can build a theory of the *isolated matter system*, i.e. of the system with a finite number of (Lagrangian) degrees of freedom. Without the possibility of defining, through observation, a trajectory $q_i = q_i(t)$, $(i = 1, \ldots, f)$, as we have seen above, there vanishes our ability to speak of a well defined, localized, isolated matter system, whose *identity* is the very "condicio sine qua non" for defining its degrees of freedom, in spite of QM' s pretenses. Thus we

may conclude that the impossibility of identifying a trajectory, inherent in quantum kinematics, engenders the impossibility of uniquely identifying the system with the finite number of degrees of freedom, whose evolution, classically, that very trajectory describes, and therefore to define the very starting point of QM. We shall see that the only way to understand the quantum-mechanical computations and to justify their undeniable success is to realize that QM is nothing but an approximation of QFT in the limit when its "field densities" become extremely small.¶ Naturally, the eclipse of QM as a basic physical theory will also mean the eclipse of the Copenhagen's view as a new, unsavory image of the physical world, upon which so much bad philosophy has been recently built.

2.4 Quantum fields are the only realistic physical objects

Let us now look in detail into the kinematical structure of QFT. In order to be able to concentrate on the fundamental conceptual aspects I shall try to keep the descriptive level as simple as possible, thus foregoing any pretense of rigour and completeness which, again, cannot be the main actual concern of the realist. The quantum field we shall focus on is the self-interacting complex scalar field $\Psi(x,t)$ in one-dimensional space,‖ whose non-relativistic Lagrangian density can be written (from now on we shall use units where $\hbar = c = 1$)

$$\mathcal{L} = i\Psi^\star \frac{\partial}{\partial t}\Psi + \frac{1}{2m}\Psi^\star \frac{\partial^2}{\partial x^2}\Psi - V(\Psi^\star\Psi)\,, \tag{2.4.1}$$

where m is the mass of the field and V is an appropriate function of $\Psi^\star\Psi$, the potential. The quantization of $\mathbf{\Psi}$ can be accomplished in full analogy with the "first quantization" of QM, i.e. one identifies the momentum density

$$\Pi(x,t) = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = i\Psi^\star(x,t)\,, \tag{2.4.2}$$

thus $\Psi(x,t)$ and $\Pi(x,t)$, for each x (the space variable), are the variables of the classical, ∞-dimensional PS, which get promoted to the field operators $\mathbf{\Psi}(x,t)$ and $\mathbf{\Pi}(x,t)$ acting on a suitable Hilbert space of physical states and

¶How small is small will be made clear in due time.

‖For a good introduction to QFT and its relevant applications to modern physics see J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields, [Bjorken and Drell (1965)] For some applications to condensed matter physics see G. Preparata, QED Coherence in Matter (World Scientific, Singapore, 1995).

obeying the equal-time CCR's

$$[\mathbf{\Psi}(x,t),\mathbf{\Pi}(x',t)] = i\delta(x-x'),\tag{2.4.3}$$

which by virtue of (2.4.2) reduces to

$$[\mathbf{\Psi}(x,t),\mathbf{\Psi}^\dagger(x',t)] = \delta(x-x').\tag{2.4.4}$$

In order to keep the mathematical complexity to a minimum level, we quantize the fields in the interval $-\frac{L}{2} < x < \frac{L}{2}$, so that the Fourier theorem allows us to decompose the field into plane-waves, yielding

$$\mathbf{\Psi}(x,t) = \frac{1}{\sqrt{L}}\sum_p \mathbf{a}_p(t)e^{ipx},\tag{2.4.5}$$

where $p = n\frac{2\pi}{L}$, n integer. It is immediate to derive from (2.4.4) the corresponding equal-time commutation relations for the quantum amplitudes $\mathbf{a}_p(t)$ and $\mathbf{a}^\dagger{}_p(t)$:

$$[\mathbf{a}_p(t),\mathbf{a}^\dagger{}_{p'}(t)] = \delta_{pp'}.\tag{2.4.6}$$

It is quite remarkable that the CCR's (2.4.6) are capable to give us a complete picture of the extremely complex kinematical structure of the Hilbert space of the states of the quantum field $\mathbf{\Psi}$, the Fock space. Indeed (2.4.6) teaches us that we can associate to each independent wave-mode p of the field a one-dimensional quantum system, an oscillator, whose $\mathbf{p}$ and $\mathbf{q}$ are related to $\mathbf{a}$ and $\mathbf{a}^\dagger$ through

$$\mathbf{a} = \frac{1}{\sqrt{2}}(\mathbf{p}-i\mathbf{q}), \qquad \mathbf{a}^\dagger = \frac{1}{\sqrt{2}}(\mathbf{p}+i\mathbf{q}),\tag{2.4.7}$$

whose CCR

$$[\mathbf{q},\mathbf{p}] = i,\tag{2.4.8}$$

readily implies

$$[\mathbf{a},\mathbf{a}^\dagger] = 1.\tag{2.4.9}$$

It is well known that a complete orthonormal basis of the Hilbert space of the quantum oscillator is provided by the states (n is a non-negative integer)

$$|n\rangle = \frac{1}{\sqrt{n!}}(\mathbf{a}^\dagger)^n|0\rangle,\tag{2.4.10}$$

where the ground state $|0\rangle$ is the unique state that is annihilated by the operator $\mathbf{a}$, i.e. $\mathbf{a}|0\rangle = 0$. It is easy to check that $\{|n\rangle\}$ is the complete set of the eigenvectors of the number-operator $\mathbf{N} = \mathbf{a}^\dagger \mathbf{a}$ and $\{n\}$ the set of its eigenvalues.

The kinematical independence of oscillators pertaining to different p's implies that the Hilbert space of the physical states is nothing but the ∞-dimensional tensor product of the ∞-dimensional Hilbert spaces for each wave-mode p. An enormously complex structure which, surprisingly, admits a particularly simple description. But in order to have a better physical understanding of the Fock space let us consider the free-field limit, i.e. the case $\mathbf{V}(\mathbf{\Psi}^\star\mathbf{\Psi}) = 0$. The classical Hamiltonian is now given by

$$H = \int dx(\Pi\dot{\Psi} - \mathcal{L}) = \int dx \left[-\frac{1}{2m}\Psi^\star \frac{\partial^2}{\partial x^2}\Psi\right], \qquad (2.4.11)$$

and to convert it into the quantum operator $\mathbf{H}$, we must prescribe the ordering of the non-commuting operators $\mathbf{\Psi}$ and $\mathbf{\Psi}^\star$. General consistency considerations demand that when going from the classical functions to the quantum operators one makes the substitution:

$$\Psi^\star\Psi \to \frac{1}{2}(\mathbf{\Psi}^\dagger\mathbf{\Psi} + \mathbf{\Psi}\mathbf{\Psi}^\dagger). \qquad (2.4.12)$$

Thus (2.4.11) yields for the quantum Hamiltonian

$$\mathbf{H} = \int dx \left\{-\frac{1}{4m}\left[\mathbf{\Psi}^\dagger \frac{\partial^2}{\partial x^2}\mathbf{\Psi} + \left(\frac{\partial^2}{\partial x^2}\mathbf{\Psi}\right)\mathbf{\Psi}^\dagger\right]\right\}, \qquad (2.4.13)$$

which through the Fourier decomposition (2.4.5) can be expressed as:

$$H = \sum_p (\frac{p^2}{2m})(\mathbf{a}^\dagger{}_p \mathbf{a}_p + \frac{1}{2}) \qquad (2.4.14)$$

where use has been made of the commutators (2.4.6). As for the momentum of the field, it is given by the operator

$$\begin{aligned}\mathbf{P} &= \int dx \left(-\frac{i}{2}\right)\left[\mathbf{\Psi}^\dagger \frac{\partial}{\partial x}\mathbf{\Psi} + \left(\frac{\partial}{\partial x}\mathbf{\Psi}\right)\mathbf{\Psi}^\dagger\right] \\ &= \sum_p p\left(\mathbf{a}^\dagger{}_p \mathbf{a}_p + \frac{1}{2}\right). \end{aligned} \qquad (2.4.15)$$

The remarkable feature of the operators $\mathbf{H}$ and $\mathbf{P}$ is that they are *simultaneously* diagonal in the $\{n_p\}$-basis (also called number representation) of the

Fock space; and on the generic vector (the index p denotes the wave-mode)

$$|\{n_p\}\rangle = \prod_p |n_p\rangle_p\,, \tag{2.4.16}$$

the total energy is given by

$$E(\{n_p\}) = \sum_p \frac{p^2}{2m}\left(n_p + \frac{1}{2}\right); \tag{2.4.17}$$

while the total momentum is

$$P(\{n_p\}) = \sum_p p\left(n_p + \frac{1}{2}\right). \tag{2.4.18}$$

The first remarkable aspect of the above results is that the Ground State (GS), that we may also call the Vacuum, which corresponds to the wave-modes being *all* in their ground state (for which $n_p = 0$), has indeed zero momentum (due to the cancellations between opposite momenta) but its energy is given by:

$$E_{GS} = \frac{1}{2}\sum_p \left(\frac{p^2}{2m}\right), \tag{2.4.19}$$

a divergent quantity. Here we make our first encounter with a very unpalatable aspect of the present formulation of QFT that is usually referred to as "ultraviolet divergence", and has substantially contributed to confine the use and study of QFT to a rather restricted number of physicists, mainly interested in particle physics. We note that the divergence of E_{GS} stems from the sum extending over an infinite numbers of wave-modes p. The necessity of summing over an infinite range is dictated upon us by our (tacit) assumption that space is continuous, i.e. that we can distinguish two different space-points even when their distance is arbitrarily small. Naturally there is no experimental evidence for such an assumption, the "continuity" of space being presently tested down to space-distances of about 10^{-16} cm only, while there is a strong theoretical evidence, based on the theory of Quantum Gravity (QG),** that at the Planck distance ($a_P \simeq 10^{-33}$ cm) such continuity gets lost due to the quantum fluctuations of the gravitational field. As a result a natural momentum cutoff $P_P = \frac{\pi}{a_P} \simeq 10^{19}$ GeV is seen to emerge, delivering us a finite and finally realistic QFT.

**I am referring to a recent work [Cacciatori *et. al* (1998)] which shows that a most likely physical realization of QG requires that space–time be a kind of a "foam" with fulls and voids of the size of a_P, the Planck distance.

However, even without a serious analysis of the structure of space at very short distances, the pragmatic attitude of the quantum field theorist so far has been to ignore the ultraviolet divergence of E_{GS} for it is in no fashion observable,[††] the only observable energies being the energy differences between the GS and all other states of the Fock space, which can be written:

$$\Delta E(\{n_p\}) = E(\{n_p\}) - E_{GS} = \sum_p n_p \left(\frac{p^2}{2m}\right), \tag{2.4.20}$$

and are thus finite. We leave here this rather loaded problematics, by noting that in the last few years the worst clouds on its real physical meaning have been clearing. For the purposes of this Essay it is only important to note that with QFT empty space, the Vacuum, ceases to be the negation of being, like in Classical Physics, to become the Ground State, the state of minimum energy, of *all* the quantum fields that exist in nature. In other (metaphysical) words, in the quantum world the Vacuum does not precede creation but is, actually, a fundamental piece of it.

Getting back to Eqs. (2.4.17) and (2.4.18), we are finally able to recognize the "physical content" of the quantum field in the free (isolated) limit. Indeed, the generic state (2.4.16) is seen to correspond to a quantum field configuration where the wave-mode p is "populated" by n_p "quanta", whose kinematical behaviour is identical to that of a system of n_p non-interacting classical particles, of momentum p and (kinetic) energy $\frac{p^2}{2m}$, and thus of mass m. Naturally to the individual "quanta", which are observed in the interactions of the field with external agents (other fields), we *cannot* attribute any autonomous physical reality, i.e a physical Hilbert space of their own, as QM pretends. Their reality begins and ends with that of their field, whose interactions with the other fields they happen, under certain conditions, to mediate. In particular, one should note, the space-coordinate does not appear in the theory as a quantum observable, like the momentum (2.4.15), but rather as a simple space-label of the amplitudes of the quantum field: in QFT the notions of quantum localization and separation are thus totally extraneous to the reality of the fields for all times in all space.

[††]This statement is actually incorrect, for it only refers to the energy exchanges between the field and any observer or other fields, but it neglects the interaction with the gravitational field, which is coupled to the energy–momentum tensor that in the GS, according to (2.4.20), is divergent. However, even in the finite version of the QFT, this fact poses severe problems to the large scale structure of space–time by predicting a *cosmological constant* some 120 orders of magnitude larger than its present limit.

To conclude, the notions of localization and separation, that realism demands of any physical theory of the "quantum", and that are so patently violated by both QM *and* Nature, imply that in any realistic physical theory of the "quanta" their clearcut objective definition must be structurally and logically *impossible*. This situation happens to hold in QFT, where localization and separation are (approximate) physical properties of the measuring apparatus, and are in no way intimately connected to the reality of the field. Thus, as far as we know today, quantum fields are the only theoretical constructs that conform to a realistic picture of the world.

Chapter 3

Dynamics: The Laws of Evolution of Physical Reality

In the Greek language $\delta\acute{\upsilon}\nu\alpha\mu\iota\varsigma$, $\delta\upsilon\nu\acute{\alpha}\mu\epsilon\omega\varsigma$ means force, power. Thus in modern physics Dynamics has come to denote the science of physical processes, the quantitative description of the time evolution of physical systems under the action of external forces. It is interesting that in Aristotle's (as well as in the Medieval) view of the physical world no distinction could be made between Dynamics and Kinematics, for motion was nothing but the process by which the bodies of the sublunar world reach, once perturbed, their "natural place". We owe it to the intellectual "heroism" of Galileo Galilei, at the beginning of the XVII century, that instantaneous motion should not be viewed as a *process* but rather as a *state* of a material system, and that Dynamics deals quantitatively with the physical causes that *change* one state into another as time flows. Thus we may rightly assert that modern science came into existence only when a clear understanding was achieved of the fundamental distinction between Kinematics and Dynamics, a distinction which has profoundly shaped both classical and quantum physics.

3.1 The Hamilton–Lagrange theory of classical dynamics

This section does not intend at all to give even a sketchy account of the way the Analytical Mechanics of Hamilton and Lagrange is built from a set of fundamental principles.* It aims, rather, at a recapitulation of ideas and mathematical formulae of classical dynamics, which will prove necessary

*For this the reader is invited to consult classical textbooks such as E. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies [Whittaker (1970)].

when their extension (and metamorphosis) to the quantum world will be discussed.

We have already seen that the classical kinematics of a Lagrangian system of f degrees of freedom (the limit $f \to \infty$ corresponding to a field system) is fully described by its f Lagrangian coordinates:

$$q_1(t), q_2(t), \ldots, q_f(t).$$

The Dynamics of such system is solved once the differential equations are found that determine uniquely, given appropriate initial conditions, the trajectories $q_i(t)$ $(i = 1, \ldots, f)$. The general method to derive such equations of motion has its focal points in two fundamental notions: the *Lagrange function*

$$L(q_i, \dot{q}_i, t), \tag{3.1.1}$$

and the *Action*

$$A = \int_{t_i}^{t_f} dt L(q_i, \dot{q}_i, t), \tag{3.1.2}$$

which is a "functional" in the function-space of the trajectories $q_i(t)$, that start at time t_i and end at time t_f.

Given the Lagrange-function (or Lagrangian) $L(q_i, \dot{q}_i, t)$, which thus embodies all the dynamics of the system, the equations of motion are those which solve the "isoperimetric problem" of finding the trajectory that renders the action A stationary, while keeping the boundary values

$$q_k(t_i) = q_{k_i}, \qquad q_k(t_f) = q_{k_f} \tag{3.1.3}$$

fixed. The calculus of variations shows that the sought equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, \tag{3.1.4}$$

the Lagrange equations: a set of f differential equations for $q_k(t)$, whose highest time-derivatives are of the second order and which, according to a well-known theorem of analysis, admit a unique solution both for the boundary conditions (3.1.3), and for the more usual ones:

$$q_k(t_i) = q_{k_i}, \qquad \dot{q}_k(t_i) = \dot{q}_{k_i}. \tag{3.1.5}$$

Equation (3.1.4) with the boundary conditions (3.1.3) or (3.1.5), give mathematical substance to the complete determinism of classical dynamics, and are the tools that the celebrated Laplace demon utilizes, once having

determined the form of the Lagrangian L, to predict with perfect certainty the future. The Lagrangian formulation, whose fundamental variables, out of which the Lagrangian is constructed, are q_k, and $\dot{q_k}$ can be *transformed* into the Hamiltonian one, whose variables are q_k and p_k, the coordinates of classical Phase Space (PS), by means of what is known as a Legendre transformations. Let us first define the momentum

$$p_k = \frac{\partial L}{\partial \dot{q_k}}(q_k, \dot{q_k}, t) \tag{3.1.6}$$

"conjugate to q_k". The Hamilton function, or Hamiltonian, is then given by

$$H(q_k, p_k; t) = \sum_{k=1}^{f} p_k \dot{q_k} - L(q_i, \dot{q_i}; t) \, . \tag{3.1.7}$$

It is clear from (3.1.6) that one can solve all $\dot{q_k}$ (at fixed time t) in terms of q_k and p_k and, once this is done, the Hamiltonian (3.1.7) becomes a function of the variables q_k and p_k. A very important construct in Hamiltonian mechanics is that of *Poisson's bracket* of two *observables*, which are well defined functions of the "canonical" variables q_k and p_k. Let $A(q_k, p_k; t)$ and $B(q_k, p_k; t)$ be such observables at a given time t, the Poisson's bracket of A and B is then defined as

$$\{A, B\} = \sum_{k=1}^{f} \left(\frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial q_k} \frac{\partial A}{\partial p_k} \right) , \tag{3.1.8}$$

which is cleary antisymmetric for the interchange $A \leftrightarrow B$, i.e

$$\{A, B\} = -\{B, A\} \, . \tag{3.1.9}$$

By substituting in (3.1.8) q_i for A and p_j for B we find

$$\{q_i, p_j\} = \delta_{ij} \, . \tag{3.1.10}$$

Assuming, for simplicity, that neither L nor H explicitly depend on t, the Lagrange equations (3.1.4), by virtue of (3.1.6) and (3.1.7), become

$$\dot{p_k} = -\frac{\partial H}{\partial q_k} , \tag{3.1.11}$$

while differentiating (3.1.7) with respect to p_k we obtain:

$$\dot{q_k} = \frac{\partial H}{\partial p_k} \, . \tag{3.1.12}$$

Equations (3.1.11) and (3.1.12) are the Hamiltonian equations of the dynamical system, whose canonical variables are q_k and p_k. Unlike the Lagrange equations, second order differential equations in the variables q_k, they are first order in the canonical variables. It is important to note that Eqs. (3.1.11) and (3.1.12) can be given an interesting form in terms of Poisson's brackets. In fact it is easy to see that according to the definition (3.1.8)

$$\dot{p_k} = \{p_k, H\}, \tag{3.1.13}$$

and

$$\dot{q_k} = \{q_k, H\}, \tag{3.1.14}$$

coincide with (3.1.11) and (3.1.12) respectively. Thus the time-derivatives of both p_k and q_k are simply given by their Poisson's bracket with the Hamiltonian. It is remarkable that in Hamiltonian mechanics this is true of *any* physical observable $O(q_k, p_k)$. Indeed we have

$$\begin{aligned} \frac{d}{dt}O(q_k, p_k) &= \sum_{k=1}^{f}\left(\frac{\partial O}{\partial q_k}\dot{q}_k + \frac{\partial O}{\partial p_k}\dot{p}_k\right) \\ &= \sum_{k=1}^{f}\left(\frac{\partial O}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial q_k}\frac{\partial O}{\partial p_k}\right) = \{O, H\}. \end{aligned} \tag{3.1.15}$$

Setting $O = H$, from the antisymmetry (3.1.9) of the Poisson's bracket, we derive that for an isolated system

$$\frac{d}{dt}H = 0, \tag{3.1.16}$$

i.e. the Hamiltonian is a constant of motion or said differently, the energy is conserved. The canonical variables (q_k, p_k) can be subjected to any set of continuous transformations

$$\begin{aligned} Q_k &= Q_k(q, p), \\ P_k &= P_k(q, p), \end{aligned} \tag{3.1.17}$$

where the functions Q_k and P_k are sufficiently well behaved so that the system (3.1.17) can be inverted. Among the (non-linear) transformations of the type (3.1.17) a special role is played in Analytical Mechanics by the so-called "canonical transformations", which leave the Hamilton system

invariant, i.e. setting

$$H'(Q,P) = H(q,p)\,, \tag{3.1.18}$$

for a "canonical transformation" it so happens that

$$\begin{aligned} \dot{P}_k &= -\frac{\partial H'}{\partial Q_k}\,, \\ \dot{Q}_k &= \frac{\partial H'}{\partial P_k}\,. \end{aligned} \tag{3.1.19}$$

It is easy to show that if (3.1.19) is satisfied the Poisson's brackets for the new variables

$$\{Q_h, P_k\} == \sum_{j=1}^{f} \left(\frac{\partial Q_h}{\partial q_j}\frac{\partial P_k}{\partial p_j} - \frac{\partial P_k}{\partial q_j}\frac{\partial q_h}{\partial p_j} \right) = \delta_{hk} \tag{3.1.20}$$

are also left invariant. Thus the canonical transformations form that group of (non-linear) transformations of the classical PS that leaves the Hamiltonian dynamics invariant in form.

3.2 The Hamiltonian operator: the generator of quantum dynamics

It was P.A.M Dirac in 1927 who discovered that once the quantum kinematics was correctly described, the quantum dynamics could be directly and simply inferred from the Hamiltonian classical dynamics. The remarkable observation of the young Dirac was that the Poisson's bracket could be set in direct correspondence with (equal time) commutators of the corresponding observables, i.e. between the Poisson's bracket $\{A, B\}$ of two classical observable and the commutator $[\mathbf{A}, \mathbf{B}]$ of their corresponding Hermitian operators, there exists the relation:

$$i\hbar\{A, B\} \to [\mathbf{A}, \mathbf{B}]\,, \tag{3.2.1}$$

which is fully compatible with the antisymmetry of the commutator, and its vanishing in the classical limit $\hbar \to 0$.

The first successful testing ground of (3.2.1) is evidently the CCR's, which directly follow from the canonical Poisson's bracket (3.1.10). In this way Dirac's identification produces in an astonishingly simple fashion the basic layout of quantum kinematics; but what is even more remarkable is that also from his discovery quantum dynamics emerges in a perfectly

natural way. Indeed, from Eq. (3.1.15) the time derivative of the quantum observable $\mathbf{O}$ is simply given by

$$i\hbar\frac{d}{dt}\mathbf{O}(t) = [\mathbf{O}(t), \mathbf{H}]\,, \tag{3.2.2}$$

which is also known as Heisenberg's equation of motion. Equation (3.2.2) exhibits in a particularly transparent way the fundamental role that the Hamiltonian operator plays in determining the dynamical evolution of the quantum system. For a Hamiltonian explicitly independent of time the solution of (3.2.2) is totally straightforward, namely

$$\mathbf{O}(t) = \exp^{\frac{i}{\hbar}\mathbf{H}t}\,\mathbf{O}(0)\exp^{-\frac{i}{\hbar}\mathbf{H}t}\,, \tag{3.2.3}$$

showing that the time evolution of the Hilbert space of the physical states of the quantum system is uniquely determined by the transformation

$$U(t) = \exp^{-\frac{i}{\hbar}\mathbf{H}t}\,, \tag{3.2.4}$$

obeying the unitarity relations:

$$U(t)U^{\dagger}(t) = U^{\dagger}(t)U(t) = 1\,, \tag{3.2.5}$$

by virtue of the Hermiticity of the Hamiltonian operator. As a result any physical state evolves in time in a *purely deterministic* way according to the equation:

$$|\psi, t\rangle = \exp^{-\frac{i}{\hbar}\mathbf{H}t}\,|\psi, 0\rangle \tag{3.2.6}$$

which, due to the unitarity of $\exp^{-\frac{i}{\hbar}\mathbf{H}t}$, remains normalized at all times t.

The fundamental fact that the Hamiltonian is the generator of the unitary transformations of the Hilbert space of the physical states associated with the dynamical evolution of the quantum system, puts the complete set of the eigenvectors of $\mathbf{H}$, the energy eigenvectors, in a special and privileged position. Let us in fact take an energy eigenvector $|E, 0\rangle$, with energy eigenvalue E. According to (3.2.6) one has

$$|E, t\rangle = \exp^{-\frac{i}{\hbar}Et}\,|E, 0\rangle\,, \tag{3.2.7}$$

i.e. with the flowing of time the change of $|E, 0\rangle$ consists only in the multiplication by the phase factor $\exp -\frac{i}{\hbar}Et$. Such states , the "stationary states", are the only ones in the Hilbert space that do not experience a "deformation" during their time evolution, and are thus the good candidates to describe the states of the quantum system between perturbations, including measurements. Differentiating (3.2.6) with respect to time allows us to

express quantum dynamics in the very familiar form of the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi, t\rangle = \mathbf{H}|\psi, t\rangle\,, \tag{3.2.8}$$

the starting point of all quantum mechanical calculations. Finally a word about the quantum analogs of the canonical transformations, which turn out to be simply the totality of unitary transformations. For equal time canonical commutation relations are clearly left unchanged by any such transformation V, and

$$V^\dagger \exp^{-\frac{i}{\hbar}\mathbf{H}t}\, V = \exp^{-\frac{i}{\hbar}V^\dagger \mathbf{H} V t}\,, \tag{3.2.9}$$

as can easily be checked through the following elementary argument:

$$\begin{aligned} V^\dagger \exp^{-\frac{i}{\hbar}\mathbf{H}t}\, V &= V^\dagger \left[\sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}t\right)^n \frac{\mathbf{H}^n}{n!}\right] V \\ &= \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}t\right)^n \frac{(V^\dagger \mathbf{H} V)^n}{n!} = \exp^{-\frac{i}{\hbar}V^\dagger \mathbf{H} V t}\,. \end{aligned} \tag{3.2.10}$$

3.3 The Path Integral (PI): classical trajectories and Quantum Physics

Thanks to Dirac [Dirac (1933)] and later Richard Feynman [Feynman (1948)] that we possess today a very powerful and illuminating tool — the Path Integral — to fully appreciate the conceptual steps involved in the transition from classical to quantum physics, with respect to both the kinematical and the dynamical aspects. The Path Integral (PI) proves also very effective in giving straightforward and intuitive solutions to a few quantum mechanical problems, where the operator formalism in a Hilbert space tends to obscure the physical meaning of the calculations. The main aim of the PI approach is to establish a connection between the transition amplitudes of quantum dynamics and the trajectories of classical dynamics. In order to keep the problem at its simplest level let us work again with a single Lagrangian degree of freedom, and call $\mathbf{H}(\mathbf{Q}, \mathbf{P})$ the Hamiltonian operator. The quantum dynamical problem is completely solved if we know the transition amplitudes

$$\langle q_f, t_f | q_i, t_i\rangle = \langle q_f | \exp^{-\frac{i}{\hbar}\mathbf{H}(t_f - t_i)} |q_i\rangle\,, \tag{3.3.1}$$

between any eigenstate $|q_i\rangle$ at the initial time t_i and any eigenstate $|q_f\rangle$ at the final time t_f of the operator $\mathbf{Q}$. For, calling

$$|t_f\rangle = \int dq_f |q_f\rangle \psi(q_f, t_f)\,, \tag{3.3.2}$$

the state evolved at the time t_f from

$$|t_i\rangle = \int dq_i |q_i\rangle \psi(q_i, t_i) \tag{3.3.3}$$

the state at the initial time t_i, one clearly has:

$$\psi(q_f, t_f) = \int dq_i \langle q_f, t_f | q_i, t_i\rangle \psi(q_i, t_i)\,. \tag{3.3.4}$$

The PI approach yields a peculiar integral representation of the transition amplitude (3.3.1) in the following way. Let us divide the time interval $t_f - t_i$ in N equal subintervals of length Δt, with the intention to take the limit $\Delta t \to 0$, or $N \to \infty$ at the end. In this way the transition amplitude can be written as the quantum composition of the evolutions over N subintervals;

$$\begin{aligned}\langle q_f, t_f | q_i, t_i\rangle &= \langle q_f | \exp^{-\frac{i}{\hbar}\mathbf{H}(t_f - t_i)} |q_i\rangle \\ &= \int dq_{N-1} dq_{N-2} \cdots dq_1 \langle q_f | \exp^{-\frac{i}{\hbar}\mathbf{H}\Delta t} |q_{N-1}\rangle \\ &\qquad \times \langle q_{N-1} | \exp^{-\frac{i}{\hbar}\mathbf{H}\Delta t} |q_{N-2}\rangle \cdots \langle q_1 | \exp^{-\frac{i}{\hbar}\mathbf{H}\Delta t} |q_i\rangle\,. \end{aligned} \tag{3.3.5}$$

Let's concentrate our attention upon the matrix element

$$\langle q_{i+1} | \exp^{-\frac{i}{\hbar}\mathbf{H}\Delta t} |q_i\rangle \simeq \langle q_{i+1} | \left(1 - \frac{i}{\hbar}\mathbf{H}\Delta t\right) |q_i\rangle\,, \tag{3.3.6}$$

which, by inserting a complete set of eigenstates of $\mathbf{P}$, yields:

$$\langle q_{i+1} | \exp^{-\frac{i}{\hbar}\mathbf{H}\Delta t} |q_i\rangle = \int dp_i \langle q_{i+1} | p_i\rangle \langle p_i | \left(1 - \frac{i}{\hbar}\mathbf{H}\Delta t\right) |q_i\rangle\,. \tag{3.3.7}$$

If the ordering ambiguity of the operators $\mathbf{Q}$ and $\mathbf{P}$ in the Hamiltonian is solved by requiring that in the general algebraic expression of the quantum Hamiltonian all $\mathbf{P}$'s are to be put to the left of all $\mathbf{Q}$'s, the last matrix

element in (3.3.7) is diagonal, and one has

$$\langle p_i| \left(1 - \frac{i}{\hbar}\mathbf{H}(P,Q)\Delta t\right)|q_i\rangle = \langle p_i|q_i\rangle \left(1 - \frac{i}{\hbar}H(p_i,q_i)\Delta t\right)$$
$$= \exp^{-\frac{i}{\hbar}H(p_i,q_i)\Delta t}\langle p_i|q_i\rangle\,, \tag{3.3.8}$$

and using the explicit form (2.2.16) for $\langle p|q\rangle$ we can write

$$\langle q_f|\exp^{-\frac{i}{\hbar}\mathbf{H}(t_f-t_i)} = \int \frac{dq_{N-1}dp_{N-1}}{2\pi\hbar}\cdots\frac{dq_1dp_1}{2\pi\hbar}\frac{dp_0}{2\pi\hbar}$$
$$\exp\frac{i}{\hbar}\sum_{k=0}^{N-1}(p_k(q_{k+1}-q_k)-H(p_k,q_k)\Delta t)\,. \tag{3.3.9}$$

In view of the smallness of Δt we may set $q_{k+1} - q_k = \dot{q}_k\Delta t$. We are now in a position to take the limit $\Delta t \to 0$, thus giving a concrete meaning to the functional integral (provided it converges)

$$\int\prod_{k=1}^{N-1}\frac{dq_kdp_k}{2\pi\hbar}\frac{dp_0}{2\pi\hbar}\cdots \to \int\frac{[dq(t)dp(t)]}{2\pi\hbar}\cdots\,. \tag{3.3.10}$$

we finally get

$$\langle q_f,t_f|q_i,t_i\rangle = \int\frac{[dq(t)dp_(t)]}{2\pi\hbar}\exp\frac{i}{\hbar}\int_{t_i}^{tf}dt[p\dot{q}-H(p,q)]\,. \tag{3.3.11}$$

Equation (3.3.11) is the sought out general connection between the quantum transition amplitude $\langle q_f,t_f|q_i,t_i\rangle$ and the classical description in terms of the PS variables (q,p). It stipulates that in order to obtain the quantum transition amplitude one must "sum" over all phase-space trajectories that pass at t_i through q_i; and at t_f through q_f, assigning to each trajectory the phase

$$\Phi = \frac{1}{\hbar}\int_{t_i}^{tf}dt[p\dot{q}-H(p,q)]\,. \tag{3.3.12}$$

But a physically much more transparent representation can be obtained when the Hamiltonian depends quadratically on the momentum p, i.e. when we may write:

$$H = \frac{p^2}{2m} + V(q)\,. \tag{3.3.13}$$

When this happens the p-integration can be done analytically in the following way

$$\int \prod_{k=0}^{N-1} \frac{dp_k}{2\pi\hbar} e^{\frac{i}{\hbar}\Delta t(p_k \dot{q}_k - \frac{p_k^2}{2m})} \to \int \frac{[dp(t)]}{2\pi\hbar} e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt(p\dot{q} - \frac{p^2}{2m})},$$

which is nothing but the product of $N-1$ Gaussian integrals

$$\int \frac{dp}{2\pi\hbar} e^{\frac{i}{\hbar}\Delta t(p\dot{q} - \frac{p^2}{2m})} = \left(\frac{m}{2i\pi\hbar\Delta t}\right)^{\frac{1}{2}} \exp\left(\frac{i}{\hbar}\Delta t \frac{m\dot{q}^2}{2}\right) \tag{3.3.14}$$

Thus we arrive at the result:

$$\langle q_f, t_f | q_i, t_i \rangle = (\text{const}) \int [dq(t)] e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt[\frac{1}{2m}\dot{q}^2 - V(q)]}$$

$$= (\text{const}) \int [dq(t)] e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt L(q,\dot{q})}, \tag{3.3.15}$$

which expresses the quantum transition amplitude as a sum over classical trajectories starting at q_i and ending at q_f of the phase factors, which are just the action of the classical trajectory (3.1.2) in unit of $\hbar$. When considered from the point of view of classical dynamics the quantum evolution, embodied in the transition amplitude $\langle q_f, t_f | q_i, t_i \rangle$, now fully reveals its profound departure from the intuitive world of the classical trajectories $q(t)$. According to (3.3.15) the evolution from q_i at time t_i to q_f at time t_f does not involve a single trajectory but *the totality of trajectories* that connect the two points at the two times, each trajectory summed with the "weight" $\exp \frac{i}{\hbar} A$, determined by its classical action A. The wave-like character of the quantum evolution is thus particularly transparent, for (3.3.15) makes direct contact with the classical connection between the motion of a wave and that of a bunch of particles whose trajectories are at right angles with the wave-front. Also the idea that to each trajectory one should associate a phase related to the Action does belong to classical Hamiltonian mechanics. What is peculiarly new, however, is the appearance of Planck constant as a fundamental unit for the Action. This fact makes us finally understand quantitatively when a classical approximation to the quantum reality becomes adequate. Indeed when the Actions associated with the classical paths $q(t)$ are much larger than $\hbar$, as it happens for the motion of the center of mass of a macroscopic body, the principle of stationary phase shows that the only relevant trajectory in the PI is that for which the action

is stationary, i.e. for which

$$\delta A = 0\,, \tag{3.3.16}$$

whose solution leads, as we know, to the Lagrange equations of classical mechanics. When the Actions involved are not large with respect to $\hbar$ the quantum evolution process involves a bundle of trajectories around the classical one, in full agreement with Heisenberg's uncertainty principle, and the basic stucture of quantum kinematics.

To conclude, we may say that in QP the notion of a trajectory (or of a well defined classical field configuration $\phi(x,t)$) disappears because in its evolution a quantum system, developing from a given initial configuration to a final one, due to the fundamental quantum fluctuations, *explores* a large number of trajectories beyond the one of stationary Action, which is just the classical trajectory.

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Chapter 4

Quantum Field Theory: The Only Realistic Theory of the Quantum World

In Sec. 1 we have amply discussed the reasons why QM *cannot* be a realistic theory of the quantum world, which may be summarized in the impossibility to give an objective (i.e. independent of the subject, the observer) meaning to the key notion of wave–particle complementarity. We have also seen that the most problematic aspect of the picture of the quantum world that QM paints us is the physical nature of the quantum particle, an object that, we should be aware, is quite distinct from the "quantum" of Einstein and Planck. Whereas, in fact, the "quantum" is a particular manifestation of the associated field and does not enjoy any dynamical autonomy, the "quantum particle" is, according to QM, a well defined object, much like the Newtonian mass-point, but for the fundamental, and puzzling, difference that the very physical means to define it, by following its trajectory, is *in principle* unavailable. In this sense, we may well say that the quantum particle is a truly metaphysical object, for no unique objective physical observations exist to give it a real substance. On the other hand no such difficulties affect the notion of field, that describes in which way a given region of space differs from empty space, where any physical observation yields by definition a null result. Localization and separability, two concepts that, we have seen, haunt QM, have no fundamental relevance in field theory, for the definition of space and time belongs to the observers through their measuring apparatus (including rigid rods and clocks), and not to the object of field theory, which represents and describes the "physical condition" of the particular region of space–time the observer focusses his attention upon.

In order to understand this latter point a little better, let us direct our attention to the way in which a classical field gets measured, for instance an electric field. One moves a test charge around, in a region whose points have been previously labelled through appropriately chosen (by the

observer) cartesian coordinates, and measures the acceleration that the test charge is subject to at any particular point, which can be converted into the numerical components of a vector, the electric field. Naturally in this procedure one heavily relies on classical physics: first the test particle, that must be localized, then the measurement of acceleration, proportional to the time derivative of the momentum of the test particle at the same point. But is it really true that in order to detect and quantitatively determine the presence of the field in an appropriately small region of space–time we *need* a "classical measurement"? One may legitimately doubt this, for all we need to ascertain is the presence of the field able to detect an *energy exchange* between the field and the measuring apparatus, whose interaction with the field may be totally quantum mechanical, like it happens, for istance, in a photomultiplier. Thus we are led to identify in the process of energy-exchange between the field(s) and a general measuring apparatus localized in space and time, the fundamental means by which we reveal how space is "modified" by field. As a result the main outcome of a field measurement in an appropriately localized region of space–time turns out to be the transition of the field between two states, whose energy difference is the one that is measured in a "small" region of space, determined by the measuring apparatus. And, as we shall see, such is the way in which the "quantum" gets "distilled" from the greatly wider reality of the field.

4.1 A preliminary discussion of coherent states

As we are interested only in the main structural aspects of QFT we shall pursue our study of the simple one-dimensional field theory of Sec. 2.4. We have seen that, through the Fourier decomposition (2.4.5), the quantum field $\mathbf{\Psi}(\mathbf{x}, \mathbf{t})$ can be reduced to the ensemble of its one-dimensional quantum oscillators of amplitudes a_p. It is thus very useful to have a better grasp of the quantum mechanics of the simple one-dimensional oscillator.

Let us then get back to the quantum oscillator, a complete basis of whose Hilbert space is spanned by the states $|n\rangle$ (see Eq. (2.4.10)). It turns out to be advantageous, at this point, to spend some time to analyze in some detail the basis of eigenstates of the operator $\mathbf{a}$, satisfying the eigenvalue equation

$$\mathbf{a}|\alpha\rangle = \alpha|\alpha\rangle\,, \tag{4.1.1}$$

where α is a complex number. It is not difficult to show that*

$$|\alpha\rangle = \exp\left(-\frac{\alpha^\star\alpha}{2}\right)\sum_{n=0}^{\infty}\frac{(\alpha)^n}{\sqrt{n!}}|n\rangle = \exp\left(-\frac{\alpha^\star\alpha}{2}\right)\exp(\alpha\mathbf{a}^\dagger)|\mathbf{0}\rangle \qquad (4.1.2)$$

obeys (4.1.1) and is normalized, i.e $\langle\alpha|\alpha\rangle = 1$. Due to the non-Hermitian nature of the "creation operator" the basis $|\alpha\rangle$ is not orthonormal, for we have:

$$\begin{aligned}\langle\alpha|\alpha'\rangle &= \exp\left(\alpha'\alpha^\star - \frac{\alpha^{\star\prime}\alpha'}{2} - \frac{\alpha^\star\alpha}{2}\right)\\ &= \exp -\frac{1}{2}|\alpha-\alpha'|^2\exp -\frac{1}{2}(\alpha^{\star\prime}\alpha - \alpha^\star\alpha')\,. \end{aligned}\qquad (4.1.3)$$

The states (4.1.2) are called "coherent states" for, according to (4.1.1), the phase of the quantum amplitude is seen to tend for large amplitudes to a well defined value, the phase of the complex number α. Let us indeed define the quantum phase Φ as

$$\mathbf{a} = e^{i\Phi}(\mathbf{a}^\dagger\mathbf{a})^{\frac{1}{2}} = e^{i\Phi}\mathbf{N}^{\frac{1}{2}}\,, \qquad (4.1.4)$$

where $N = \mathbf{a}^\dagger\mathbf{a}$ is the number operator, diagonal with integer eigenvalues on the basis $|n\rangle$. It is an easy exercise to show that the canonical commutation relation $[\mathbf{a},\mathbf{a}^\dagger] = 1$ implies that

$$[\mathbf{N},\Phi] = i \qquad (4.1.5)$$

which shows that the phase operator Φ is conjugate to the number operator $\mathbf{N}$. Thus according to the Heisenberg principle

$$(\Delta\Phi)(\Delta N) \geq \frac{1}{2}\,, \qquad (4.1.6)$$

implying that in the states where there is a fixed number of "quanta" (see Sec. 2.4), i.e. $\Delta N = 0$, the phase is completely undetermined. On the other hand we compute

$$\langle\alpha|e^{i\Phi}|\alpha\rangle = \langle\alpha|\mathbf{a}\mathbf{N}^{-\frac{1}{2}}|\alpha\rangle = \sum_{n=0}^{\infty}\frac{1}{[n!(n+1)!]^{\frac{1}{2}}}\alpha|\alpha|^{2n}e^{-|\alpha|^2}\,, \qquad (4.1.7)$$

which for $|\alpha| \gg 1$ yields ($\alpha = |\alpha|e^{i\phi}$)

$$\langle\alpha|e^{i\Phi}|\alpha\rangle \approx e^{-|\alpha|^2}\alpha\frac{e^{|\alpha|^2}}{|\alpha|} = e^{i\phi} \qquad (4.1.8)$$

*This follows from the easily established identity $[\mathbf{a}, \exp\alpha\mathbf{a}^\dagger] = \alpha\exp\alpha\mathbf{a}^\dagger$.

stipulating that for $|\alpha|$ large (note that $\Delta N = |\alpha|$) the phase is well defined and equal to the phase of α. One can also easily show that the completeness relation for coherent states can be expressed as

$$1 = \sum_{n=0}^{\infty} |n\rangle\langle n| = \int \frac{d\alpha d\alpha^\star}{2\pi i} |\alpha\rangle\langle\alpha| , \tag{4.1.9}$$

which allows us to decompose any state $|\psi\rangle$ of the quantum oscillator as:

$$|\psi\rangle = \int \frac{d\alpha d\alpha^\star}{2\pi i} \langle\alpha|\psi\rangle|\alpha\rangle . \tag{4.1.10}$$

Thus to the generic state $|\psi\rangle$ one can associate a holomorphic function $\psi(\alpha)$ of the complex variable α:

$$|\psi\rangle \to \psi(\alpha) = e^{\frac{\alpha\alpha^\star}{2}} \langle\psi|\alpha\rangle , \tag{4.1.11}$$

in such representation

$$\mathbf{a}|\psi\rangle \to \frac{d}{d\alpha}\psi(\alpha) , \tag{4.1.12}$$

while

$$\mathbf{a}^\dagger|\psi\rangle \to \alpha\psi(\alpha) , \tag{4.1.13}$$

and the scalar product is given by

$$\langle\phi|\psi\rangle = \int \frac{d\alpha d\alpha^\star}{2\pi i} e^{-\alpha\alpha^\star} \psi^\star(\alpha)\phi(\alpha) . \tag{4.1.14}$$

It is very easy now to compute the spectrum of the number operator $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}$. Indeed the eigenvalue equation is

$$\mathbf{a}^\dagger\mathbf{a}|\nu\rangle = \nu|\nu\rangle , \tag{4.1.15}$$

which is equivalent to

$$\alpha\frac{d}{d\alpha}\phi_\nu(\alpha) = \nu\phi_\nu(\alpha) \tag{4.1.16}$$

whose solution is obviously $\phi_\nu(\alpha) = c_\nu\alpha^\nu$. Now the holomorphy of $\phi_\nu(\alpha)$ restricts ν to the set of non-negative integers $n = 0, 1, \dots$, and the normalization condition (4.1.3) fixes $c_n = \frac{1}{\sqrt{n!}}$.

The PI representation in this basis takes on a particularly simple form

$$\langle\alpha_f, t_f|\alpha_i, t_i\rangle = \int \langle\alpha_f|e^{-iH\Delta t}|\alpha_{N-1}\rangle\langle\alpha_{N-1}|e^{-iH\Delta t}|\alpha_{N-2}\rangle$$

$$\cdots\langle\alpha_1|e^{-iH\Delta t}|\alpha_i\rangle \prod_{k=1}^{N-1} \frac{d\alpha_k^\star d\alpha_k}{2\pi i}\,. \tag{4.1.17}$$

Prescribing the ordering of $\mathbf{a}$ and $\mathbf{a}^\dagger$ so that all $\mathbf{a}^\dagger$ are to the left of all $\mathbf{a}$:

$$\begin{aligned}\langle\alpha_{k+1}|e^{-iH\Delta t}|\alpha_k\rangle &= e^{-iH(\alpha_{k+1}^\star,\alpha_k)\Delta t}\langle\alpha_{k+1}|\alpha_k\rangle \\ &= e^{-\frac{\alpha_{k+1}^\star\alpha_{k+1}}{2}} e^{-\frac{\alpha_k^\star\alpha_k}{2}} e^{\alpha_{k+1}^\star\alpha_k} e^{-i\Delta t H(\alpha_{k+1}^\star,\alpha_k)}\,,\end{aligned} \tag{4.1.18}$$

thus

$$\begin{aligned}\langle\alpha_f, t_f|\alpha_i, t_i\rangle &= \int \prod_{k=1}^{N-1} \frac{d\alpha_k^\star d\alpha_k}{2\pi i} \exp i \sum_k (\alpha_k^\star i\dot\alpha_k - H(\alpha_k^\star, \alpha_k))\Delta t \\ &\xrightarrow[\Delta t\to 0]{} \int \frac{[d\alpha^\star(t)d\alpha(t)]}{2\pi i} \exp i \int_{t_i}^{t_f} dt(\alpha^\star(t)i\dot\alpha(t) \\ &\quad - H(\alpha^\star, \alpha))\,.\end{aligned} \tag{4.1.19}$$

In the "free" theory, where $H(\alpha^\star, \alpha) = \omega\alpha^\star\alpha$, the PI provides a particularly simple solution of the dynamics of the oscillator: by going to the new variables

$$\beta(t) = \alpha(t)e^{i\omega(t-t_i)}$$

$$\begin{aligned}\langle\alpha_f, t_f|\alpha_i, t_i\rangle &= \int \frac{[d\beta^\star(t)d\beta(t)]}{2\pi i} \exp i \int_{t_i}^{t_f} dt(\beta^\star(t)i\dot\beta(t) \\ &= \langle\beta_f|\beta_i\rangle = \langle\beta_f|\alpha_i\rangle = \langle\alpha_f e^{i\omega(t_f-t_i)}|\alpha_i\rangle \\ &= \langle\alpha_f|e^{-i\omega(t_f-t_i)}\alpha_i\rangle\,,\end{aligned} \tag{4.1.20}$$

where the last scalar products have the form given in (4.1.3). This last result has a very simple interpretation: the dynamical evolution of a coherent state in the "free" theory goes through coherent states whose amplitudes evolve as

$$\alpha(t) = e^{-i\omega(t-t_i)}\alpha_i \tag{4.1.21}$$

It is also interesting to note that the latter result coincides with the classical one. Indeed the principle of stationary action of classical physics, in this case applied to (4.1.19), takes the form

$$\delta \int_{t_i}^{t_f} dt[\alpha^\star(t) i\dot{\alpha}(t) - \omega\alpha^\star(t)\alpha(t)] = 0\,, \tag{4.1.22}$$

whose Lagrange equation is

$$i\dot{\alpha}(t) = \omega\alpha(t)\,, \tag{4.1.23}$$

whose solution is just (4.1.21).

We may thus rightly conclude that the coherent states represent the "best" quantum approximation of classical physics.

As a last "exercise" in quantum oscillators, we shall consider the problem solved long ago by N.N. Bogoliubov: the diagonalization of the Hamiltonian for two coupled oscillators (λ positive)

$$\mathbf{H} = (\omega + \lambda)(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{a}_2^\dagger \mathbf{a}_2) + \lambda(\mathbf{a}_1 \mathbf{a}_2 + \mathbf{a}_1^\dagger \mathbf{a}_2^\dagger)\,, \tag{4.1.24}$$

which can be solved by finding first two new oscillator amplitudes:

$$\mathbf{A}_1 = \alpha\mathbf{a}_1 + \beta\mathbf{a}_2^\dagger\,, \tag{4.1.25}$$

$$\mathbf{A}_2 = \gamma\mathbf{a}_2 + \delta\mathbf{a}_1^\dagger\,, \tag{4.1.26}$$

such that

$$[\mathbf{A}_1, \mathbf{A}_1^\dagger] = [\mathbf{A}_2, \mathbf{A}_2^\dagger] = \mathbf{1}\,, \qquad [\mathbf{A}_1, \mathbf{A}_2] = [\mathbf{A}_1^\dagger, \mathbf{A}_2^\dagger] = 0 \tag{4.1.27}$$

It is easy to show that the conditions (4.1.27) are satisfied if

$$\alpha = \gamma = \cosh\theta \quad \beta = \delta = \sinh\theta\,. \tag{4.1.28}$$

The solution is then obtained by finding the value of θ such that

$$\mathbf{H} = \nu(\mathbf{A}_1^\dagger \mathbf{A}_1 + \mathbf{A}_2^\dagger \mathbf{A}_2) + \mu\,. \tag{4.1.29}$$

A rather simple, though tedious, calculation yields:

$$\tanh 2\theta = \frac{\lambda}{\omega + \lambda} \tag{4.1.30}$$

$$\nu = E = \sqrt{\omega^2 + 2\lambda\omega} \tag{4.1.31}$$

$$\mu = -\frac{\lambda^2}{2}\frac{1}{\omega + \lambda + \sqrt{(\omega+\lambda)^2 - \lambda^2}} < 0\,. \tag{4.1.32}$$

The spectrum of (4.1.29) is

$$E_{n_1 n_2} = (n_1 + n_2)E + \mu\,, \tag{4.1.33}$$

corresponding to "quanta" of energy $E > \omega$, the perturbative energy, and a ground state, $n_1 = n_2 = 0$, whose energy μ is negative, i.e. lower than the perturbative ground state ($\lambda \to 0$). One may wonder how does the state $|0\rangle_\theta$, that is annihilated by $\mathbf{A_1}$ and $\mathbf{A_2}$, look in terms of the eigenstates of the number operators $\mathbf{a}_1^\dagger \mathbf{a}_1$ and $\mathbf{a}_2^\dagger \mathbf{a}_2$, the answer is

$$|0\rangle_\theta = \frac{1}{\cosh\theta}\sum_{k=0}^{\infty}(-1)^k \frac{(\tanh\theta)^k}{k!}(\mathbf{a}_1^\dagger \mathbf{a}_2^\dagger)^k|0\rangle\,, \tag{4.1.34}$$

showing that the new ground state $|0\rangle_\theta$ is a *coherent* superposition of states with an indefinite number of pairs of quanta of both oscillators.

4.2 The Vacuum, the template of physical reality

Given a QFT, through its Lagrangian density and the CCR's, such as in Sec. 3.4, we are confronted with the task of understanding its structural and dynamical content. How does one begin? The answer is simple: find the Vacuum, the Ground State (GS), the state of minimum energy. Why must one begin with the Vacuum? The answer is again simple: due to the fundamentally fluctuating nature of the quantum fields, if the Universe is open (as we shall always assume) any quantum state in a *finite* region of space will decay after an appropriate time, depending on the interactions among different fields (including those associated with the observers), to the state of minimum energy, the only stable state.

Thus finding the Vacuum is the preliminary step to figure out in which way that particular region of space, we focus our attention upon, will react to our measuring devices, thereby informing us on the quantum states of the fields present in that region. As we shall show below, the structure of the Vacuum, which would seem by definition unknowable, determines in a fundamental way the kind of physical excitations, or "quanta", our measuring devices detect or excite out of the quantum fields, through their energy negotiations with the fields themselves. In other words, the Vacuum turns out to be a kind of "template" of the physical reality of the quantum fields.

For definiteness' sake, let us write the classical Hamiltonian of our one-dimensional field theory as:

$$\begin{aligned} H = & -\frac{1}{2m}\int \Psi^\star(x,t)\frac{\partial^2}{\partial x^2}\Psi(x,t)dx \\ & +\frac{1}{2}\int dxdyV(x-y)\Psi^\star(x,t)\Psi(x,t)\Psi^\star(y,t)\Psi(y,t) \\ & +g\int \Psi^\star(x,t)\Psi(x,t)dx\,, \end{aligned} \tag{4.2.1}$$

where g is a real parameter, which can be either positive or negative. The quartic term in (4.2.1) describes the field self-interaction through the potential $V(x-y)$, proportional to the *field densities* $\Psi^\star\Psi$ at the two (arbitrary) space points x and y.

Going to the quantum field $\mathbf{\Psi}$, i.e. quantizing the classical theory as outlined in Sec. 2.4, we can express the quantum Hamiltonian as:

$$\begin{aligned} \mathbf{H} = & \frac{1}{2}\sum_p\left(g+\frac{p^2}{2m}\right)+\sum_p\left(g+\frac{p^2}{2m}\right)\mathbf{a}^\dagger{}_p\mathbf{a}_p \\ & +\frac{1}{2L}\sum_{p_1,p_2,q}V_q\mathbf{a}^\dagger{}_{p_1+q}\mathbf{a}^\dagger{}_{p_2-q}\mathbf{a}_{p_1}\mathbf{a}_{p_2}\,, \end{aligned} \tag{4.2.2}$$

where the Fourier transform of the potential $V(x)$ is defined by:

$$V(x)=\frac{1}{L}\sum_q e^{-iqx}V_q\,.$$

We have seen that for $V_q\to 0$ the Vacuum, the Ground State (GS) of the QFT, is given by the tensor product of the ground states $|0\rangle_p$ of the mode oscillators $\mathbf{a}_p$. Such perturbative GS (PGS) is the starting point of most field theoretical calculations, that treat the quartic term as a perturbation, which seems reasonable if V_q is "small". However it may so happen that one can approximate the interaction with other fields by a term $g\,\mathbf{\Psi}^\star\mathbf{\Psi}$, with $g<0$. In this case the PGS loses any meaning and the GS, the Vacuum, changes its structure completely. Let us see why and how.

Let's assume that the vacuum expectation value of the field $\mathbf{\Psi}$, which vanishes in the PGS, is different from zero, i.e.

$$\langle GS|\mathbf{\Psi}(x,t)|GS\rangle=\frac{\alpha}{\sqrt{L}}\,, \tag{4.2.3}$$

with α a complex number. From (2.4.5) this corresponds to shifting the a_p's in (4.2.2) as

$$a_p \to a_p + \delta_{p_0}\alpha\,. \tag{4.2.4}$$

As a result we are led to a new (effective) Hamiltonian:

$$\begin{aligned}\mathbf{H}_{eff} &= E_{PGS} + g\alpha^\star\alpha + \sum_{p\neq 0}\left(g + \frac{p^2}{2m}\right)\mathbf{a}^\dagger{}_p\mathbf{a}_p \\ &+ \frac{1}{2L}V_0(\alpha^\star\alpha)^2 + \frac{V_0}{L}\alpha^\star\alpha\sum_{p\neq 0}\mathbf{a}^\dagger{}_p\mathbf{a}_p \\ &+ \frac{1}{2L}\sum_{p\neq 0}V_p[2\alpha^\star\alpha\mathbf{a}^\dagger{}_p\mathbf{a}_p + \alpha^{\star 2}\mathbf{a}_p\mathbf{a}_{-p} + \alpha^2\mathbf{a}^\dagger{}_p\mathbf{a}^\dagger{}_{-p}] + \cdots\end{aligned} \tag{4.2.5}$$

where the dots denote the terms cubic and quartic in the oscillator amplitudes $\mathbf{a}_p$ and $\mathbf{a}^\dagger{}_p$, which can be treated perturbatively if the V_p's are small enough. The minimization of the GS energy yields,

$$g\alpha + \frac{V_0}{L}\alpha^\star\alpha \quad \alpha = 0\,, \tag{4.2.6}$$

admitting a nontrivial solution if $V_0 > 0$ and $g < 0$, i.e. if the energy of interaction with other fields is negative. In this case, we have a minimum for

$$\alpha^\star\alpha = -\frac{gL}{V_0}\,. \tag{4.2.7}$$

By a simple phase transformation we can replace α by a real number a, and write

$$\begin{aligned}H_{eff} &= E_{PGS} - \frac{g^2L}{2V_0} + \sum_{p\neq 0}\left[\frac{p^2}{2m} + \lambda_p\right]\mathbf{a}^\dagger{}_p\mathbf{a}_p \\ &+ \frac{1}{2}\sum_p \lambda_p(\mathbf{a}_p\mathbf{a}_{-p} + \mathbf{a}^\dagger{}_p\mathbf{a}^\dagger{}_{-p}) + \cdots\,,\end{aligned} \tag{4.2.8}$$

where

$$\lambda_p = \frac{-V_p}{V_0}g\,, \tag{4.2.9}$$

which, at least for small p's, is positive. In Sec. 4.1 we have seen how to diagonalize the quadratic part of the effective Hamiltonian. The Bogoliubov

transformation thus yields

$$\tanh 2\theta_p = \frac{\frac{-V_p}{V_0} g}{\frac{p^2}{2m} - g\frac{V_p}{V_0}}, \tag{4.2.10}$$

$$E_p = \sqrt{\left(\frac{p^2}{2m}\right)^2 - 2g\frac{p^2}{2m}\left(\frac{V_p}{V_0}\right)}, \tag{4.2.11}$$

and

$$\mu_p = -\frac{g^2\left(\frac{-V_p}{V_0}\right)^2}{2\left[\frac{p^2}{2m} - g\frac{V_p}{V_0} + E_p\right]}. \tag{4.2.12}$$

The model we have just analyzed, though very simple, gives us a rather vivid picture of the massive rearrangement of the degrees of freedom of the quantum field that occurs in presence of a negative ($g < 0$) energy term, i.e of an instability, stemming from the interaction with other fields.† The non-vanishing of the expectation value (4.2.3) of the field Ψ in the non-perturbative GS has the following fundamental consequences:

- it lowers the energy of the PGS by the quantity:

$$\Delta E = E_{GS} - E_{PGS} = \sum_p \mu_p - \frac{g^2}{2V_0}L, \tag{4.2.13}$$

  implying that the PGS is unstable and cannot represent a decent approximation of the physics of the quantum field;
- the particle spectrum drastically changes from the free one

$$\epsilon_p|_{PGS} = \frac{p^2}{2m}, \tag{4.2.14}$$

  being given instead by

$$E_p = \left[\left(\frac{p^2}{2m}\right)^2 - g\frac{p^2}{m}\frac{V_p}{V_0}\right]^{\frac{1}{2}} \xrightarrow[p\to 0]{} \tag{4.2.15}$$

$$\xrightarrow[p\to 0]{} (-g)^{\frac{1}{2}}\frac{p}{\sqrt{m}}. \tag{4.2.16}$$

†This is just what happens to the quark fields in interaction with the colour gauge fields of QCD, leading to their "confinement". See G. Preparata, Il Nuovo Cimento, [Preparata (1986)].

We may thus conclude that the instability of the PGS changes completely the structure of the quanta of the field Ψ, which do not have any more the character of (non-relativistic) point particles of mass m, but appear as collective excitations, similar to "phonons", with sound velocity $v_s = (\frac{-g}{m})^{\frac{1}{2}}$.

We end this section with the analysis of the GS of our QFT in the condensed matter limit, i.e when in our interval L the number of particles is fixed and equal to N. In this case the expectation value on the GS of the number operator

$$\hat{\mathbf{N}} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \boldsymbol{\Psi}^\dagger(\mathbf{x},\mathbf{t})\boldsymbol{\Psi}(\mathbf{x},\mathbf{t}) = \sum_p \mathbf{a}^\dagger{}_p \mathbf{a}_p \tag{4.2.17}$$

is given by

$$\langle GS|\hat{\mathbf{N}}|GS\rangle = N. \tag{4.2.18}$$

The minimization equation (4.2.6) is now replaced by

$$\alpha^\star \alpha = N,\ g = -\frac{V_0}{L} N\,. \tag{4.2.19}$$

As a result the effective Hamiltonian is

$$\begin{aligned} H_{eff} &= E_{PGS} - \frac{V_0}{2}\frac{N^2}{L} + \sum_{p\neq 0}\left(\frac{N}{L}V_p + \frac{p^2}{2m}\right)\mathbf{a}^\dagger{}_p\mathbf{a}_p \\ &+ \frac{N}{2L}\sum_p V_p[\mathbf{a}_p\mathbf{a}_{-p} + \mathbf{a}^\dagger{}_p\mathbf{a}^\dagger{}_{-p}] + \cdots\,. \end{aligned} \tag{4.2.20}$$

which differs from (4.2.8) in the ground state, energy being now

$$E_{GS} = E_{PGS} - \frac{V_0}{2}\frac{N^2}{L}\,, \tag{4.2.21}$$

and in the value of λ_p, now given by

$$\lambda_p = \frac{N}{L}V_p\,, \tag{4.2.22}$$

leading to a "phonon" spectrum with sound velocity

$$v_s = \left(\frac{V_p}{m}\frac{N}{L}\right)^{\frac{1}{2}}\,. \tag{4.2.23}$$

4.3 The "classical" limit of QFT: the emergence of coherence

When analyzed in terms of its mode oscillators, the classical limit of the quantum field is seen to involve states $|n\rangle_p$ with very large occupation number n. This is a consequence of the well-known "principle of correspondence" introduced in QM by N. Bohr, according to which in the high-n limit the quantum behavior should merge with the classical one.

Let's consider again our simple model in the condensed matter limit, which forces the field to evolve in the subsector of its Hilbert space where the number operator

$$\hat{\mathbf{N}} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \mathbf{\Psi}^\dagger(x,t)\mathbf{\Psi}(x,t) \tag{4.3.1}$$

is diagonal and its eigenvalue is equal to N. This immediately suggests to "normalize" the field ψ as

$$\mathbf{\Psi}(x,t) = N^{\frac{1}{2}}\mathbf{\Phi}(x,t)\,, \tag{4.3.2}$$

where the field $\mathbf{\Phi}$ is normalized in a Schrödinger-like way, i.e. to one.

Let us now consider the PI representation of the transition amplitude $\langle [\alpha_{p_f}], t_f | [\alpha_{p_i}], t_i\rangle$ between the coherent states of the field oscillators $\mathbf{a}_p$ at time t_i and those at time t_f. A straightforward generalization of (4.1.17) allows us to write

$$\begin{aligned}\langle [\alpha_{p_f}], t_f | [\alpha_{p_i}], t_i\rangle = &\int \prod_p [d\alpha_p(t) d\alpha_p^\star(t)] \\ &\times \exp i \int_{t_i}^{t_f} dt \left[\sum_p \alpha_p^\star i\dot{\alpha}_p - H(\alpha_p, \alpha_p^\star)\right].\end{aligned} \tag{4.3.3}$$

Normalizing the field according to (4.3.2), i.e. going from the variables α_p to the new variables

$$\beta_p = N^{-\frac{1}{2}}\alpha_p\,, \tag{4.3.4}$$

the PI can be rewritten

$$\begin{aligned}\langle [\beta_{p_f}], t_f | [\beta_{p_i}], t_i\rangle = &\int \prod_p [N d\beta_p(t) d\beta_p^\star(t)] \\ &\times \exp iN \int_{t_i}^{t_f} dt \left[\sum_p \beta_p^\star i\dot{\beta}_p - H_N(\beta_p, \beta_p^\star)\right],\end{aligned} \tag{4.3.5}$$

where from (4.2.2)

$$H_N = \sum_p \left(g + \frac{p^2}{2m}\right)\beta_p^\star\beta_p + \frac{1}{2}\left(\frac{N}{L}\right)\sum_{p_1,p_2,q} V_q\beta^\star_{p_1+q}\beta^\star_{p_2-q}\beta_{p_1}\beta_{p_2}\,, \quad (4.3.6)$$

which in the fixed density $(\frac{N}{L})$ limit is in fact N-independent. The interesting aspect of the representation (4.3.5) is the explicit appearance of the large number N in the same position of $\frac{1}{\hbar}$ (which however in our units system has been set to one, and has thus dropped out of sight). Formally, then, the condensed matter limit $(N \to \infty)$ is completely equivalent to the classical limit $(\hbar \to 0)$ of the diluted system.

This important observation immediately suggests to us the simple solution of the PI: the field evolves along a classical trajectory, solution of the stationary phase equations:

$$i\dot{\beta}_p = \left(g + \frac{p^2}{2m}\right)\beta_p + \left(\frac{N}{L}\right)\sum_{p_1,q} V_q\beta^\star_{p_1+q}\beta_{p-q}\beta_{p_1}\,, \quad (4.3.7)$$

around which it performs quantum fluctuations of amplitudes $O(\frac{1}{\sqrt{N}})$.

We have seen in Sec. 4.1 that for a quantum system the classical trajectory gives (approximately) at each time the numerical values of the phases of the coherent state in which it finds itself at that time. Let us now analyze what kind of informations an observer may acquire on the system by measuring for instance the charge (if the field's charge is e) in a small interval Δa around the point x_0. The observed "observable" is thus

$$\mathbf{Q}\left(x_0 - \frac{\Delta a}{2}, x_0 + \frac{\Delta a}{2}\right) = e\int_{x_0-\frac{\Delta a}{2}}^{x_0+\frac{\Delta a}{2}} dx\,\mathbf{\Psi}^\dagger(x,t)\mathbf{\Psi}(x,t)\,. \quad (4.3.8)$$

Let's assume for simplicity that only the mode with $p = 0$ is occupied and that the charge "leaks" out of the interval $(-\frac{L}{2}, \frac{L}{2})$, so that the "classical trajectory" is given by

$$\alpha_p(t) = \delta_{p_0} N^{\frac{1}{2}} e^{-\frac{\lambda t}{2}} \quad (4.3.9)$$

The expectation value of the charge on the coherent state with amplitude (4.3.9) is:

$$\bar{Q} = \langle\alpha(t)|\mathbf{Q}|\alpha(t)\rangle = e\left(\frac{\Delta a}{L}\right)|\alpha(t)|^2\,, \quad (4.3.10)$$

while its dispersion is readily evaluated as

$$\Delta Q = e\left(\frac{\Delta a}{L}\right)|\alpha(t)| \tag{4.3.11}$$

As long as $|\alpha(t)|^2 \gg 1$, i.e. $t \ll \frac{1}{\lambda}\ell nN$, the regime is essentially classical, for the quantum fluctuations of the charge measurement (4.3.11) are much smaller than the expectation value (4.3.10). Note, however, that according to quantum principles the measuring device necessarily registers the charge value en, with a probability distribution peaked at $\bar{n} = \frac{\bar{Q}}{e}$ and with a dispersion $\Delta\bar{n} = \frac{\Delta\bar{Q}}{e}$. This very important consequence of the quantum principles can be understood from the fundamental hypothesis that any interaction between the field and the measuring device *cannot* involve a non-integer number of quanta, the elementary field excitations over the GS, whose charge is e.

It is only when $|\alpha(t)|^2\frac{\Delta a}{L}$ is $O(1)$, i.e. when the field is in a very "dilute" state, that quantum reality begins to shine with its discontinuities and probabilities, the latter being intimately and inextricably tied to the former. In this limit the value $(\frac{\bar{Q}}{e}) < 1$ represents only the probability that a charge measurement in the interval Δa yields the result e, and the expectation value of the operator $\boldsymbol{\Psi}^\dagger(x,t)\boldsymbol{\Psi}(x,t)$ can be interpreted as the uniform probability density to find a single charge in the interval Δa around x_0, just like the square of the single particle Schrödinger wave-function in ordinary QM. More of this in the next section.

We end this section by analyzing in the condensed matter limit a very important and simple model: the Dicke system of N two-level atoms, the basic description of Laser physics. In our simple one-dimensional field theory its Hamiltonian is

$$H = E_1\int_{-\frac{L}{2}}^{\frac{L}{2}} \boldsymbol{\Psi}_1^\dagger\boldsymbol{\Psi}_1 dx + E_2\int_{-\frac{L}{2}}^{\frac{L}{2}} \boldsymbol{\Psi}_2^\dagger\boldsymbol{\Psi}_2 dx + (E_2 - E_1)\mathbf{a}^\dagger\mathbf{a}$$
$$+\frac{g}{\sqrt{L}}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx(\boldsymbol{\Psi}_2^\dagger\boldsymbol{\Psi}_1\mathbf{a} + \boldsymbol{\Psi}_1^\dagger\boldsymbol{\Psi}_2\mathbf{a}^\dagger)\,, \tag{4.3.12}$$

and the number operator:

$$\hat{\mathbf{N}} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx(\boldsymbol{\Psi}_1^\dagger\boldsymbol{\Psi}_1 + \boldsymbol{\Psi}_2^\dagger\boldsymbol{\Psi}_2)\,. \tag{4.3.13}$$

The physical meaning of the model is quite clear: $\mathbf{\Psi_1}$ and $\mathbf{\Psi_2}$ denote the matter fields, whose quanta are atoms in the energy state E_1 and E_2 respectively, $\mathbf{a}$ is the amplitude of the transverse e.m. field with frequency $\omega = (E_2 - E_1)$ in resonance with the atomic transition $1 \leftrightarrow 2$, and the fourth term represents the electromagnetic coupling of the two atomic levels. In (4.3.12) and (4.3.13) we have chosen the length of our interval L to be smaller than the wave-length $\lambda = \frac{2\pi}{\omega}$ of the resonant electromagnetic mode so as to be able to neglect the spatial variation of both the matter and the e.m. field. This restriction, which appears as a choice of convenience, has an interesting physical meaning and motivation, for it implies that the spatial region, in which the matter and the e.m. fields are defined, gets naturally partitioned in "Coherence Domains" (CD) of the size λ, where their amplitudes are essentially constant, that rapid variations of the fields and the quantum and thermal fluctuations being confined to their boundaries.[‡] Thus the analysis that follows focusses on the physics of a single CD.

Another important "caveat" about the Hamiltonian (4.3.12) is that the term $\omega\mathbf{a}^\dagger\mathbf{a}$ of the Dicke model is an oversimplification of the full Hamiltonian, based on the so-called "slowly varying envelope approximation", which assumes that, writing the e.m. field amplitude, $a = \alpha(t)e^{-i\omega t}$, the time-dependence of the envelope $\alpha(t)$ can be neglected. If one does not make this approximation which, as we shall see, clearly breaks down in the highly condensed matter limit, the one-mode Lagrangian that appears in the phase of the PI is[§]

$$L_{em} = \frac{i}{2}[\alpha^\star\dot{\alpha} - \dot{\alpha}^\star\alpha] + \frac{1}{2\omega}\dot{\alpha}^\star\dot{\alpha}\,. \tag{4.3.14}$$

The PI representation of the transition amplitude $\langle f|i\rangle$ can then be written ($\omega = (E_2 - E_1)$)

$$\langle f|i\rangle = \int \left[\frac{da_1(t)da_1^\star(t)}{2\pi i}\frac{da_2(t)da_2^\star(t)}{2\pi i}\frac{d\alpha d\alpha^\star(t)}{2\pi i}\right]$$
$$\exp i\int_{t_i}^{t_f}\left\{a_1^\star i\dot{a}_1 + a_2^\star i\dot{a}_2 - E_1a_1^\star a_1 - E_2a_2^\star a_2 + \frac{i}{2}[\alpha^\star\dot{\alpha} - \alpha\dot{\alpha}^\star]\right.$$
$$\left. + \frac{1}{2\omega}\dot{\alpha}\dot{\alpha}^\star + \frac{g}{\sqrt{L}}(a_2^\star a_1\alpha e^{-i\omega t} + a_1^\star a_2\alpha^\star e^{i\omega t})\right\}. \tag{4.3.15}$$

[‡]The physical relevance of CD's in condensed matter is discussed in the book of Ref. [Preparata (1995)].

[§]As shown in the above mentioned book.

The condensed matter limit stems from the condition

$$\langle \mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{a}_2^\dagger \mathbf{a}_2 \rangle = N \tag{4.3.16}$$

which advises us, as in (4.3.2), to "rescale" the amplitudes as

$$\begin{aligned} b_{1,2} &= a_{1,2} N^{-\frac{1}{2}}\,, \\ \beta &= \alpha N^{-\frac{1}{2}}\,. \end{aligned} \tag{4.3.17}$$

As a result the PI (4.3.15) gets transformed into

$$\begin{aligned} \langle f|i\rangle \sim \int [db_1 db_1^\star db_2 db_2^\star d\beta d\beta^\star] \exp iN \int_{t_i}^{t_f} dt \\ \times \Bigg\{ b_1^\star i\dot{b}_1 + b_2^\star i\dot{b}_2 - E_1 b_1 b_1{}^\star - E_2 b_2 b_2^\star + \frac{i}{2}[\beta^\star \dot{\beta} - \beta\dot{\beta}^\star] + \frac{1}{2\omega}\dot{\beta}\dot{\beta}^\star \\ + g\sqrt{\frac{N}{L}}(b_2^\star b_1 \beta e^{-i\omega t} + b_1^\star b_2 \beta^\star e^{+i\omega t}) \Bigg\} \end{aligned} \tag{4.3.18}$$

which can be further simplified through the "interaction representation" transformation:

$$\begin{aligned} b_1 &\to e^{-iE_1 t}\beta_1 \\ b_2 &\to e^{-iE_2 t}\beta_2 \end{aligned}$$

yielding the PI Action

$$\begin{aligned} A_{PI} = N \int_{t_i}^{t_f} dt \Bigg[\beta_1^\star i\dot{\beta}_1 + \beta_2^\star i\dot{\beta}_2 + \frac{i}{2}(\beta^\star \dot{\beta} - \beta\dot{\beta}^\star) + \frac{1}{2\omega}\dot{\beta}^\star \dot{\beta} \\ + g\sqrt{\frac{N}{L}}(\beta_2^\star \beta_1 \beta + \beta_1^\star \beta_2 \beta^\star) \Bigg] \end{aligned} \tag{4.3.19}$$

Apart from the factor N, which as we have already seen, controls the classical limit, A_{PI} is remarkable for two more reasons: the disappearance of the rapidly varying phase factor $e^{\pm i\omega t}$, and the large amplification by $\sqrt{N}$ of the coupling term, which shows its crucial importance in the high density $\sqrt{\frac{N}{L}}$ limit. The "classical" equations of motion are now obtained by the stationarity of the action with respect to the variations $\delta\beta_{1,2}^\star$ and $\delta\beta^\star$ [the

variations of $\delta\beta_{1,2}$ and $\delta\beta$ yielding the complex conjugate equations]:

$$i\dot{\beta}_1 + g\sqrt{\frac{N}{L}}\beta_2\beta^\star = 0\,, \tag{4.3.19a}$$

$$i\dot{\beta}_2 + g\sqrt{\frac{N}{L}}\beta_1\beta = 0\,, \tag{4.3.19b}$$

$$-\frac{1}{2\omega}\ddot{\beta} + i\dot{\beta} + g\sqrt{\frac{N}{L}}\beta_1^\star\beta_2 = 0\,, \tag{4.3.19c}$$

with the condition [see (4.3.16)]

$$|\beta_1|^2 + |\beta_2|^2 = 1 \tag{4.3.20}$$

I shall not analyze in detail the interesting consequences of the differential system (4.3.19), which has been amply studied in the mentioned book "QED Coherence in the Matter". Here I will only demonstrate a fundamental consequence of this simple, but extremely realistic model of the quantum physics of matter coupled to the e.m. field: the so-called "Superradiant Phase Transition" (SPT).¶ Let us first rewrite the system in terms of the adimensional time $\tau = \omega t$;

$$i\dot{\beta}_1 = -\frac{g}{\omega}\sqrt{\frac{N}{L}}\beta^\star\beta_2\,, \tag{4.3.21a}$$

$$i\dot{\beta}_2 = -\frac{g}{\omega}\sqrt{\frac{N}{L}}\beta\beta_1\,, \tag{4.3.21b}$$

$$-\frac{1}{2}\ddot{\beta} + i\dot{\beta} = -\frac{g}{\omega}\sqrt{\frac{N}{L}}\beta_1^\star\beta_2\,. \tag{4.3.21c}$$

Suppose that at $t = 0$ the system is in the perturbative ground state (PGS), as is universally assumed in today's condensed matter physics. We have

$$\beta_1(0) = 1 + O\left(\frac{1}{\sqrt{N}}\right), \quad \beta_2(0) = O\left(\frac{1}{\sqrt{N}}\right), \quad \beta(0) = O\left(\frac{1}{\sqrt{N}}\right), \tag{4.3.23}$$

the question we wish to give an answer to is: will the configuration (4.3.23) remain at all times? We are interested in the evolution of the system (4.3.21)

¶This remarkable fact, fundamental for our understanding of condensed matter physics, was discovered in this model in the '70's by K.Hepp and E. Lieb [Hepp and Lieb (1973)], but its significance was discarded due to the erroneous opinion that the model violated electrodynamic gauge-invariance. For a discussion of this gigantic blunder see the mentioned book.

in the neighbourhood of $\tau = 0$, i.e. when to a good approximation $\beta_1 \simeq 1$. The differential system simplifies as

$$i\dot{\beta}_2 = -\frac{g}{\omega}\sqrt{\frac{N}{L}}\beta\,, \tag{4.3.22a}$$

$$-\frac{1}{2}\ddot{\beta} + i\dot{\beta} = \frac{-g}{\omega}\sqrt{\frac{N}{L}}\beta_2\,, \tag{4.3.22b}$$

which is equivalent to ($G = \frac{g}{\omega}\sqrt{\frac{N}{L}}$)

$$-\frac{1}{2}\frac{d^3}{d\tau^3}\beta + i\frac{d^2}{d\tau^2}\beta + iG^2\beta = 0\,, \tag{4.3.24}$$

with the initial conditions:

$$\beta(0) = O\left(\frac{1}{\sqrt{N}}\right), \quad \dot{\beta}(0) = O\left(\frac{1}{\sqrt{N}}\right).$$

The linear nature of the differential equation (4.3.24) assures us that both β_2 and β remain of $O\left(\frac{1}{\sqrt{N}}\right)$ if and only if the algebriac equation:

$$\frac{p^3}{2} - p^2 + G^2 = 0 \tag{4.3.25}$$

has no complex roots. This condition is violated when

$$G^2 = \frac{g^2}{\omega^2}\frac{N}{L} > \frac{16}{27}\,, \tag{4.3.26}$$

implying that, given g and ω, there is a critical density $\left(\frac{N}{L}\right)_c = \frac{16}{27}\frac{\omega^2}{g^2}$ above which the system will "run away" from the perturbative configuration (4.3.23). Where to? The answer is again quite easy, one can show that the real GS, which we may call Coherent Ground State (CGS) and which obeys the system (4.3.21) with the condition (4.3.20), is given by:

$$\beta_1 = \cos\theta e^{i\theta_1(\tau)}, \quad \beta_2 = \sin\theta e^{i\theta_2(\tau)}, \quad \beta = Ae^{i\phi(\tau)}\,, \tag{4.3.27}$$

with $\theta_{1,2}$ and ϕ linear function of τ, satisfying the condition

$$\theta_1 - \theta_2 - \phi = -\frac{\pi}{2}\,. \tag{4.3.28}$$

What is most remarkable about the CGS is the phase-locking (4.3.28) between the matter and the e.m. fields, presenting us a vivid picture of matter coherently and collectively oscillating between its two atomic levels

in tune with a coherent e.m. field which, unlike what happens in the Laser, gets trapped in the matter.

Finally a word about Lasers: it should be entirely evident that lasing corresponds to a completely different configuration. Indeed, by means of the population inversion achieved by the pumping mechanism, both $\beta_1(0)$ and $\beta_2(0)$ are $0(1)$ and the short-time behaviour of the e.m. amplitude is always of the "run-away" type corresponding to the Laser's "switch-on", as the reader may easily ascertain.

4.4 The "quantum-mechanical" limit of QFT: The Schrödinger wave-function

In this section we shall study QFT, again in the simple one-dimensional model, in a limit which lies at the opposite end of the classical one, namely when the field is very *diluted*, i.e. it evolves in a subspace of the Hilbert space of the eigenvectors of the number operator $\hat{\mathbf{N}}$ in the interval L, that have small eigenvalues.

The cursory discussion of the *dilute* limit of the last section advises us to concentrate our attention upon:

$$\begin{aligned}\rho_s(x_1, x_2, \ldots, x_N, t) \\ = {}_N\langle s|\Psi^\dagger(x_1,t)\Psi(x_1,t)\cdots\Psi^\dagger(x_N,t)\Psi(x_N,t)|s\rangle_N\,,\end{aligned} \tag{4.4.1}$$

which represents the equal-time correlation of N density measurements at the *different*[||] (disjoint) points $x_1, x_2, \ldots, x_N$, on the state of the field Ψ $|s\rangle_N$, eigenvector of the number operator with eigenvalue N (in the interval L). The points $x_1, x_2, \ldots, x_N$ being disjoint, we may write:

$$\begin{aligned}\rho_s(x_1, x_2, \ldots, x_N, t) \\ = {}_N\langle s|\Psi^\dagger(x_1,t)\cdots\Psi^\dagger(x_N,t)\Psi(x_N,t)\cdots\Psi(x_1,t)|s\rangle_N\,,\end{aligned} \tag{4.4.2}$$

symmetric under any permutation of the points $x_1, x_2, \ldots, x_N$. By inserting a complete set of intermediate states $|k\rangle$, eigenstates of the number operator

[||]Please note that the disjointness condition on the space points is appropriate for describing the quantum measurements of number density upon an "isolated" system of N quanta.

with (integer) eigenvalue k, we have

$$\rho_s(x_1, x_2, \ldots, x_N, t) = \sum_k {}_N\langle s|\Psi^\dagger(x_1,t)\cdots\Psi^\dagger(x_N,t)|k\rangle$$
$$\langle k|\Psi(x_N,t)\cdots\Psi(x_1,t)|s\rangle_N \,. \qquad (4.4.3)$$

From the canonical commutation relations (2.4.4) it is straightforward to show that

$$[\hat{\mathbf{N}}, \Psi(x_N,t)\cdots\Psi(x_1,t)] = -N\Psi(x_N,t)\cdots\Psi(x_1,t)\,. \qquad (4.4.4)$$

Applying this identity to the generic matrix element of (4.4.3)

$$\begin{aligned}&\langle k|[\hat{\mathbf{N}}, \Psi(x_N,t)\cdots\Psi(x_1,t)]|s\rangle_N \\ &= -N\langle k|\Psi(x_N,t)\cdots\Psi(x_1,t)|s\rangle_N \\ &= (k-N)\langle k|\Psi(x_N,t)\cdots\Psi(x_1,t)|s\rangle_N \,,\end{aligned} \qquad (4.4.5)$$

we learn that the only non-zero matrix element in the sum (4.4.3) corresponds to the Vacuum state $|0\rangle$, allowing us to write ρ_s in the factorized form

$$\rho_s(x_1, x_2, \ldots, x_N, t) = N!|\psi_s(x_1, x_2, \ldots, x_N, t)|^2 \,, \qquad (4.4.6)$$

where

$$\psi_s(x_1, x_2, \ldots, x_N, t) = \frac{1}{\sqrt{N!}}\langle 0|\Psi(x_N,t)\cdots\Psi(x_1,t)|s\rangle_N \,. \qquad (4.4.7)$$

We shall now show that $\psi_s(x_1, x_2, \ldots, x_N, t)$ has all the properties of the Schrödinger wave-function of the N-particle system, and therefore *is* the Schrödinger wave-function.

First of all the normalization. The disjointness condition is equivalent to expressing $\rho_s(x_1, x_2, \ldots, x_N, t)$ as the Wick-product, (: ... :) where all $\Psi^\dagger$ operators are to the left of all Ψ operators:

$$\begin{aligned}&\rho_s(x_1, x_2, \ldots, x_n, t) \\ &= {}_N\langle s| :\Psi^\dagger(x_1,t)\Psi(x_1,t)\cdots\Psi^\dagger(x_N,t)\Psi(x_N,t): |s\rangle_N \,,\end{aligned} \qquad (4.4.8)$$

and integrating over all spatial variables one has

$$\int dx_1 \cdots dx_N \; \rho_s(x_1, x_2, \ldots, x_N, t) = N! \,, \qquad (4.4.9)$$

as can be readily shown in the following way:

$$\begin{aligned}
&\int dx_1 \cdots dx_N {}_N\langle s|\boldsymbol{\Psi}^\dagger(x_1,t)\cdots\boldsymbol{\Psi}^\dagger(x_N,t)\boldsymbol{\Psi}(x_N,t)\cdots\boldsymbol{\Psi}(x_1,t)|s\rangle_N \\
&\quad = \int dx_1 \cdots dx_{N-1} {}_N\langle s|\boldsymbol{\Psi}^\dagger(x_1,t)\cdots\hat{\mathbf{N}}\cdots\boldsymbol{\Psi}(x_1,t)|s\rangle_N \\
&\quad = \int dx_1 \cdots dx_{N-1} {}_N\langle s|\boldsymbol{\Psi}^\dagger(x_1,t)\cdots\boldsymbol{\Psi}^\dagger(x_{N-1},t) \\
&\qquad \times \boldsymbol{\Psi}(x_{N-1},t)\cdots\boldsymbol{\Psi}(x_1,t)|s\rangle_N\,,
\end{aligned} \tag{4.4.10}$$

which stems from the fact that, by an argument already used, the only non-vanishing matrix element of the type

$$\langle k|\boldsymbol{\Psi}(x_{N-1},t)\cdots\boldsymbol{\Psi}(x_1,t)|s\rangle_N$$

is for $k = 1$, thus $|k\rangle$ is an eigenstate of the number operator $\hat{\mathbf{N}}$ with eigenvalue 1. By a similar argument

$$\begin{aligned}
&\int dx_1 \cdots dx_{N-1} {}_N\langle s|\boldsymbol{\Psi}^\dagger(x_1,t)\cdots\boldsymbol{\Psi}^\dagger(x_{N-1},t)\boldsymbol{\Psi}(x_{N-1},t)\cdots\boldsymbol{\Psi}(x_1,t)|s\rangle_N \\
&\quad = \int dx_1 \cdots dx_{N-2} {}_N\langle s|\boldsymbol{\Psi}^\dagger(x_1,t)\cdots\boldsymbol{\Psi}^\dagger(x_{N-2},t)\hat{\mathbf{N}} \\
&\qquad \times \boldsymbol{\Psi}(x_{N-2},t)\cdots\boldsymbol{\Psi}(x_1,t)|s\rangle_N \\
&\quad = 1.2\int dx_1 \cdots dx_{N-2} {}_N\langle s|\boldsymbol{\Psi}^\dagger(x_1,t)\cdots\boldsymbol{\Psi}^\dagger(x_{N-2},t) \\
&\qquad \times \boldsymbol{\Psi}(x_{N-2},t)\cdots\boldsymbol{\Psi}(x_1,t)|s\rangle_N\,,
\end{aligned} \tag{4.4.11}$$

and continuing the reduction until the last variable x_1, one clearly gets (4.4.9), which through the definition (4.4.5) coincides with the normalization condition of the N-particle Schrödinger wave-function:

$$\int |\psi_s(x_1,x_2,\ldots,x_N,t)|^2 dx_1\cdots dx_N = 1\,. \tag{4.4.12}$$

The N-particle Schrödinger equation easily follows from the Heisenberg equation of motion (3.2.2), when one sets $O(t) = \boldsymbol{\Psi}(x_N,t)\cdots\boldsymbol{\Psi}(x_1,t)$. We have then (recall that in our units system $\hbar = 1$)

$$i\frac{\partial}{\partial t}[\boldsymbol{\Psi}(x_N,t)\cdots\boldsymbol{\Psi}(x_1,t)] = [\boldsymbol{\Psi}(x_N,t)\cdots\boldsymbol{\Psi}(x_1,t),\mathbf{H}] \tag{4.4.13}$$

with the Hamiltonian:

$$\mathbf{H} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \frac{1}{2m} \boldsymbol{\Psi}^\dagger(-\nabla^2)\boldsymbol{\Psi} + \frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \int_{-\frac{L}{2}}^{\frac{L}{2}} dy$$
$$\times \boldsymbol{\Psi}^\dagger(x,t)\boldsymbol{\Psi}(x,t)V(x-y)\boldsymbol{\Psi}^\dagger(y,t)\boldsymbol{\Psi}(y,t)\,. \qquad (4.4.14)$$

An easy calculation yields for the commutator in (4.4.13)

$$\begin{aligned}
&[\boldsymbol{\Psi}(x_N,t)\cdots\boldsymbol{\Psi}(x_1,t),\mathbf{H}] \\
&\quad = \sum_{k=1}^{N} -\frac{1}{2m}\frac{\partial^2}{\partial x_k^2}\boldsymbol{\Psi}(x_N,t)\cdots\boldsymbol{\Psi}(x_1,t) \\
&\qquad + \frac{1}{2}\sum_{h,k} V(x_h - x_k)\boldsymbol{\Psi}(x_N,t)\cdots\boldsymbol{\Psi}(x_1,t)\,, \qquad (4.4.15)
\end{aligned}$$

plus terms which vanish when sandwiched between $\langle 0|$ and $|s\rangle_N$. Sandwiching (4.4.13) and (4.4.15) between $\langle 0|$ and $|s\rangle_N$ and dividing by $\frac{1}{\sqrt{N!}}$ we obtain precisely the Schrödinger equation:

$$\begin{aligned}
&i\frac{\partial}{\partial t}\psi_s(x_1,x_2,\ldots,x_N,t) \\
&\quad = \mathbf{H}\psi_s(x_1,x_2,\ldots,x_N,t) \\
&\quad = \sum_{k=1}^{N} -\frac{1}{2m}\frac{\partial^2}{\partial x_k^2}\psi_s(x_1,x_2,\ldots,x_N,t) \\
&\qquad + \frac{1}{2}\sum_{h,k} V(x_h - x_k)\psi_s(x_1,x_2,\ldots,x_N,t)\,. \qquad (4.4.16)
\end{aligned}$$

To my mind, the results we have just derived lift finally the veil upon the mystery of the "unreasonable" effectiveness of QM in accounting for the reality of the microscopic world: "unreasonable" in the light of its conventionalistic nature, impermeable to any realistic interpretation, as we have argued in the first Section of this Essay. What has come out of our analysis is that QM, with its postulates and dynamical equations, is nothing but an approximation of QFT in the limit of extreme "dilution", i.e when in the finite volume, where we observe the field, the number of particles, the eigenvalue of the number operator $\hat{\mathbf{N}}$, is a small, finite number N. When this is the case, and it is obviously so for a diluted gas of atoms or molecules, QM with its formalism and dynamical equations is a rigorous

consequence of the fundamental principles of QFT. What is, fortunately, completely gone is instead the bizarre Copenhagen interpretation, hopelessly entangled with the meaningless notion of "wave-particle" duality. In particular the alleged "definitive" argument against the possibility that the Schrödinger wave-function describes a physical process in real space–time $(\vec{x}, t)$, based *on the multidimensional character of the support of the* N-particle Schrödinger wave-function (4.4.7), is seen to evaporate due to the fact that QFT now teaches us that the wave-function is nothing but the matrix element of a string of N field operators $\mathbf{\Psi}$ at N disjoint space points and at the same time between the Vacuum and a particular state of the QFT where the operator $\hat{\mathbf{N}}$, the space (which from now on resumes its three-dimensional nature) integral of the density operator $\mathbf{\Psi}^\dagger(\vec{x}, t)\mathbf{\Psi}(\vec{x}, t)$, is diagonal and its eigenvalue is equal to N. In this case the square of the normalized wave-function [See Eq. (4.4.9)] is an appropriately normalized correlation function among the measurement of the field density at N disjoint points of *real* space at the *real* time t. And such correlation function has exactly the meaning that QM attributes to it. But where QM is now radically supplanted is in the fact that the space coordinates $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_N}$ in no way can be attributed to the N indistinguishable quantum particles, dynamical degrees of freedom of each particle, but are in a one-to-one correspondence with the positions in space where a possible** observer cares to put at the time t his density measuring devices. Whether the system is observed or not, the wave-function, as well as physical reality, doesn't care, it continues to develop in time according to the Schrödinger equation (4.4.16),†† until it reaches the ground state, the state of minimum energy E_0, solution of the stationary Schrödinger equation

$$E_0\psi_s(x_1, x_2, \ldots, x_N) = \mathbf{H}\psi_s(x_1, x_2, \ldots, x_N, t)\,. \qquad (4.4.17)$$

Finally a few words about "entanglement". The EPR critique not only of the Copenhagen interpretation, but of the asserted completeness of QM, is seen to boil down to the fact that the quantum mechanical wave-function $\psi(\vec{x_1}, \vec{x_2}; t)$ of a particular two-particle system does not factorize in the product of two functions $\phi(\vec{x_1}, t)\chi(\vec{x_2}, t)$, when the region where the two quanta are localized by the observation cannot be connected by a light ray, and are therefore "causally disconnected". The lack of factorization,

**It is not absolutely necessary that the observer be real, it is important that he exists as a "gedanken" observer.

††To which we need to add an appropriate dissipative term, that relaxes excitation energy to the open world.

or "entanglement" between the wave-functions of each quantum, can then be seen to pose unsurmontable problems to any realistic interpretation of QM, or, as Einstein liked to put it, to the "element of reality" requiring that space-like separated particles should appear to any observer endowed with their own properties only. On the other hand if factorization fails, and in general it does fail in QM, one finds typical correlations between the observations on the two particles.

The answer of QFT to EPR is very simple: the particles one is dealing with here would possess their own "element of reality" only if QM were a fundamental theory. The failure of QM to comply with EPR 's requirement is thus only another way to see that QM *cannot be* a fundamental theory, a conclusion we have reached "ad nauseam" in this Essay. On the other hand, in QFT the quantum particles have no independent physical reality, nor a well defined physical localization, they just correspond to the peculiarity of the localized quantum measurement, that can only reveal the presence of the quantum field in a "quantized" way, i.e through their quanta. Thus in QFT the "elements of reality" only belong to the quantum field, which is neither localized nor separable, *pace* of EPR.

Chapter 5

Final Considerations

As explained in the Introduction the motivation of this Essay has been mainly philosophical: a plea for and a contribution to the great tradition of *realism*, to which we owe the incredible rise of modern Science. I have shown that in order to secure a realistic approach to quantum physics one must reject the idea that QM is a *complete*, self-consistent theory of the quantum world, an idea that leaves us no choice but to subscribe to the conventionalism of the Copenhagen interpretation. And in the last section I have demonstrated that QM is a rigorous consequence of the corresponding QFT in the limit of "infinite dilution", where the world is only populated by a small, finite number of quanta.

I may easily imagine the "normal" scientist of our times sneering at this latter statement: Big deal! What have we gained? A prediction of new phenomena? A new set of computational rules? Only a different interpretation of the physical meaning of the mathematical symbols of the quantum mechanical formulae, which does not take us an inch ahead in our long journey through energy and matter. For such is the attitude of the positivists, who completely dominate modern science, generally uninterested in questions of principle, philosophy in short, and keenly concentrating on the "practical" aspects of their activity, whose range is, however, rigorously constrained by a rigid "paradigm", that stands, without their knowing, on a pedestal of philosophy, including Copenhagen's brand of conventionalism.

In the history of science it is not usually recognized that all flights away from realism, beginning with Bellarmino and the Aristotelians and ending with Lorentz and Poincaré, though difficult to attack on purely logical grounds, did nevertheless do harm to the progress of science, and proved in the end their inanity. Will this not be the case of QM conventionalism as well? I surmise that the answer to this question is positive. According to

the discussion in Sec. 3.3, if one gives up a QM description of atomic and molecular matter in favour of a full QFT, as any realist should do, then an easy approach to the "classical" limit of condensed matter becomes immediately available, through simple classical equations of motion, that exhibit the possibility of a Superradiant Phase Transition (SPT). One may object that the SPT was discovered in the Dicke two-level system, where matter is treated in the fashion of QM; however, the reason why its remarkable meaning was not appreciated, and its existence forgotten, to my mind must be attributed to the mathematical difficulty of the approach followed by K. Hepp and E. Lieb. In the full QFT treatment, on the other hand, the generality of the SPT becomes at once evident, and with it its compulsory nature.

Time will tell whether the discovery of the QED coherence of condensed matter shall induce a momentous "paradigm shift" in our picture of the micro/macroscopic world, and propel humanity towards a new era of matter and energy. What we can state with confidence now is that the realism of QFT is opening our eyes to a wonderful world of phenomena, which has so far remained hidden to our view, blocked by the prejudices and the narrow expectations of a bizarre and conventionalistic *Weltanschauung.*

I wish to thank most warmly my collaborators E. Del Giudice and G. Lo Iacono for their invaluable help.

Appendix

This Appendix is devoted to a brief description of the mathematics and the formalism that lay at the basis of Quantum Physics (QP), and have been utilized throughout this Essay.

We have seen in Sec. 2 that the birth of QP has marked a distinct departure from the kinematical description of Classical Physics, where States and Observables are in one-to-one correspondence in the classical configuration space, the Phase Space. In QP it turns out to be absolutely necessary to keep the two concepts, State and Observable, distinct and to give them different, appropriate mathematical realizations. We have seen that the physical reason of this strange, revolutionary fact is to be found, as is often emphasized, in the interaction between observer and physical system, its magnitude being bounded from below by the Planck's constant $\hbar$, as expressed by Heisenberg principle.

In order to understand what kind of mathematics must be at play to realize the fundamental quantum distinction, one may recall the essential wave character of QP and the great mathematics that through the better part of the XIX Century have been developed following the luminous ideas of Joseph Fourier. As is well known, such developments culminated in the theory of Hilbert spaces and of linear operators operating upon them, to which, as we shall see in a moment, there precisely correspond the configuration space and the observables of QP respectively. Hilbert spaces and linear operators, the basic tools of modern functional analysis, are but the generalizations to infinite dimension of the more familiar algebraic structures of finite dimensional complex vector spaces and of the matrices operating on them. Thus for an easier understanding of the mathematical structure of QP we shall proceed with finite N-dimensional complex spaces, with the idea of taking the limit $N \to \infty$ at the end. Such limit poses several

subtle problems that have troubled and occupied the mathematicians for a large part of this Century, but as once reminded us Norbert Wiener, in no way this should trouble and occupy the physicist, for he has the great privilege to have Nature as a kind of custodian angel to prevent him from making mistakes and wasting his time. And, besides, it is a fact that in the mathematical description of Nature, that a physicist can make, zero and its inverse, infinity, do not really exist, for they always imply an extrapolation into uncharted territory.

Suppose now that our quantum system *can* be found in N different states, then the fundamental postulate of QP is that to each such state one associates a complex N-dimensional vector $\vec{v} \equiv (c_1, c_2, \ldots, c_N)$ (c_k are complex numbers) that, for reasons that soon will become apparent, Dirac baptized *ket* and represented as

$$\vec{v} \to |v\rangle \tag{A.1}$$

Before we go on, let me emphasize that the above quantum postulate is really revolutionary for it implies, according to the algebraic structure of a vector space, that if $|v_1\rangle$ and $|v_2\rangle$ are two such states ($\alpha_{1,2}$ are complex numbers)

$$(\alpha_1 c_1^{(1)} + \alpha_2 c_1^{(2)}, \ldots, \alpha_1 c_N^{(1)} + \alpha_2 c_N^{(2)}) \to \alpha_1 |v_1\rangle + \alpha_2 |v_2\rangle \tag{A.2}$$

is also a possible state, thus realizing the fundamental superposition principle.

The next assumption is that our N-dimensional complex vector space is a unitary space, i.e. given any two vectors $|v_1\rangle$ and $|v_2\rangle$, one can define the *scalar product*

$$\sum_{k=1}^{N} c_k^{(1)*} c_k^{(2)} = \langle v_1 | v_2 \rangle = \langle v_2 | v_1 \rangle^* , \tag{A.3}$$

which implies that to each vector $\vec{v}$ one associates its *dual* $\vec{v}^* \equiv (c_1^*, c_2^*, \ldots, c_N^*)$

$$\vec{v}^* \to \langle v | , \tag{A.4}$$

Dirac's *bra*, so that the scalar product according to (4) has the structure of a *bra-ket*, in English bracket. In order for the scalar product to have a physical meaning, it must have some kind of invariance, and in fact one imposes that all allowed linear transformations $U \equiv \|U_{ij}\|$ that map the

vector space onto itself leaves the scalar product (3) invariant, i.e. is unitary. In Dirac's symbolism this means that

$$\langle v_1|v_2\rangle = \langle v_1|U^\dagger U|v_2\rangle\,, \tag{A.5}$$

where $U^\dagger = \|U^*_{ji}\|$ is the Hermitian conjugate of U, which means that a unitary transformation obeys

$$UU^\dagger = U^\dagger U = \mathbf{1}\,. \tag{A.6}$$

Physically the requirement of unitarity is necessary for the probability interpretation of QP, as we shall see in a moment. For the time being the invariant scalar product allows us to *normalize* the physical states, i.e. to impose that for any physical state

$$\langle v|v\rangle = \mathbf{1}\,. \tag{A.7}$$

This requirement implies that in the general linear superposition:

$$|v\rangle = \sum_k \alpha_k|v_k\rangle \tag{A.8}$$

the α_k must satisfy

$$\langle v|v\rangle = \sum_{kh} \alpha^*_h\alpha_k\langle v_h|v_k\rangle = \mathbf{1}\,. \tag{A.9}$$

Furthermore the invariant scalar product gives us the possibility to define the (N-dimensional) *orthonormal basis* $\{|v_k\rangle\}$ for which

$$\langle v_h|v_k\rangle = \delta_{hk}\,. \tag{A.10}$$

so that any state $|v\rangle$ can be given the representation (8) with the condition

$$\sum_{k=1}^{N} \alpha^*_k\alpha_k = \mathbf{1}\,. \tag{A.11}$$

It is easy to check that given an *orthonormal basis*, any other such basis can be obtained from it by a unitary transformation. In fact let $\langle|w_k\rangle\rangle$ be such a basis, with, $|w_k\rangle = U|v_k\rangle$, then

$$\langle w_h|w_k\rangle = \langle v_h|U^\dagger U|v_k\rangle = \langle v_h|v_k\rangle = \delta_{hk}\,. \tag{A.12}$$

This much for the quantum States. Let us now turn our attention to the quantum Observables. The assumption is that to each Observable there

corresponds an Hermitian operator **O**, i.e. a linear operator such that

$$\mathbf{O}^\dagger = \mathbf{O}\,. \tag{A.13}$$

This requirement insures that its *eigenvalue spectrum* is real and its eigenvectors form an *orthonormal basis.* In fact the (normalized) eigenvector $|o\rangle$, with eigenvalue o of the Observable **O** is defined by the equation

$$\mathbf{O}|\mathbf{o}\rangle = \mathbf{o}|\mathbf{o}\rangle \tag{A.14}$$

whence

$$\langle o|\mathbf{O}|\mathbf{o}\rangle = \mathbf{o}\,, \tag{A.15}$$

$$o = \langle o|\mathbf{O}|\mathbf{o}\rangle = \langle \mathbf{o}|\mathbf{O}^\dagger|\mathbf{o}\rangle = \mathbf{o}^*\,, \tag{A.16}$$

showing that the eigenvalue is real. Thus, given two eigenvectors $|o_1\rangle$ and $|o_2\rangle$ with different eigenvalues one has

$$\mathbf{O}_{1,2} = \mathbf{o}_{1,2}|\mathbf{o}_{1,2}\rangle\,. \tag{A.17}$$

Taking

$$\langle o_2|\mathbf{O}|\mathbf{o}_1\rangle = \mathbf{o}_1\langle \mathbf{o}_2|\mathbf{o}_1\rangle = \mathbf{o}_2\langle \mathbf{o}_2|\mathbf{o}_1\rangle\,, \tag{A.18}$$

one immediately derives the orthogonality of the two eigenvectors, i.e.

$$\langle o_2|o_1\rangle = 0\,. \tag{A.19}$$

If the eigenvectors have the same (real) eigenvalue, one can by a standard procedure (Schmidt's orthogonalization) form appropriate linear combinations of the two so that the new eigenvectors are orthogonal. In fact suppose that $\langle o_2|o_1\rangle \neq 0$, then $|o_1\rangle$ and

$$|o_I\rangle = \frac{|o_2\rangle - |o_1\rangle\langle o_1|o_2\rangle}{\sqrt{1 - |\langle o_1|o_2\rangle|^2}} \tag{A.20}$$

are easily shown to be orthonormal. Thus: the eigenvectors of an Hermitian operator form an *orthonormalbasis* and the spectrum of its eigenvalues is real. This result is central for the physical interpretation of the mathematics of QP; indeed if the system is in a State described by the eigenvector $|o\rangle$, the observation (measurement) of the Observable **O** will yield for sure the value o, which can be expressed as the expectation value $\langle o|\mathbf{O}|\mathbf{o}\rangle$ of **O**

upon $|o\rangle$, as follow from (16). Let's now take a generic State $|\psi\rangle$, given by the linear superposition of the *orthonormal basis* of the Observable $\mathbf{O}$

$$|\psi\rangle = \sum_{k=1}^{N} c_k |o_k\rangle \,, \tag{A.21}$$

what is the meaning of the expectation value $\langle\psi|\mathbf{O}|\psi\rangle$? We have

$$\langle\psi|\mathbf{O}|\psi\rangle = \sum_{\mathbf{hk}} \langle \mathbf{o_h}|\mathbf{O}|\mathbf{o_k}\rangle \mathbf{c_h^*}\mathbf{c_k} = \sum_{\mathbf{k}} \mathbf{o_k}|\mathbf{c_k}|^2 \,, \tag{A.22}$$

i.e. its value depends upon the square modulus of the (complex) superposition coefficients c_k, whose sum is according to (11) normalized to 1. Thus it is natural, and corroborated by experiments, to interpret $|c_k|^2$ as the probability that a measurement of $\mathbf{O}$ in the state $|\psi\rangle$ yields the value o_k. In this way the expectation value of $\mathbf{O}$ upon $|\psi\rangle$ represents the average value of $\mathbf{O}$ when a large number of measurements are performed on that State. The revolutionary consequence of all this is, of course, that when the system is in the State $|\psi\rangle$, which is not an eigenvector of $\mathbf{O}$, this Observable has no definite, sharp value but only an average value (when a large number of measurements is collected), given by its expectation value (22). However a single measurement always gives some well defined value o_k, belonging to the spectrum of its eigenvalues, the probability of this happening being precisely $|c_k|^2$. This bizarre, but real fact is usually referred to as the *reduction of the wave-function* and has caused several headaches to those who wish to give a realistic interpretation of QP. We now know, as I have argued at length in this Essay, that such *reduction* is the inescapable consequence of the non-disposable presence of the observer, without whom no physical meaning can be given to the Observable itself. As emphasized in the text, this looks really weird, but such is the architecture of the Universe!

One last aspect of the general description of a quantum system is how to *label* uniquely the N *kets* of an *orthonormal basis*. We know in general that given an Hermitian operator $\mathbf{O}$, its *orthonormal basis* defines a set of subspaces of the original vector space, inside each of which its eigenvalues do not change. The problem of *liftingthedegeneracy*, i.e. of reducing each subspace to a one-dimensional one, is solved by finding a *complete set of commuting operators*, as Dirac calls it. Indeed given two Observables $\mathbf{O_1}$ and $\mathbf{O_2}$, they possess common eigenvectors if and only if they commute, i.e. if $[\mathbf{O_1}, \mathbf{O_2}] = \mathbf{0}$, as can easily be seen. Thus: let $|\alpha\rangle$ be one such eigenvector,

then

$$\mathbf{O_{1,2}}|\alpha\rangle = \mathbf{o_{1,2}}|\alpha\rangle \tag{A.23}$$

$$[\mathbf{O_1}, \mathbf{O_2}]|\alpha\rangle = (\mathbf{O_1 O_2} - \mathbf{O_2 O_1})|\alpha\rangle = (\mathbf{o_1 o_2} - \mathbf{o_2 o_1})|\alpha\rangle = \mathbf{0}\,, \tag{A.24}$$

thus

$$[\mathbf{O_1}, \mathbf{O_2}] = \mathbf{0}\,. \tag{A.25}$$

As a result we may set $|\alpha\rangle = |o_1 o_2\rangle$, and if the *orthonormal basis* defined by the commuting $\mathbf{O_{1,2}}$ does not *lift the degeneracy*, one continues until the commuting Observables $\mathbf{O_1}, \mathbf{O_2}, \ldots, \mathbf{O_N}$ do so. At this point we are guaranteed that any other commuting Observable must be a linear combination of the *complete set*, and the orthonormal *kets* $|o_1, o_2, \ldots, o_n\rangle$ are thus uniquely *labeled.*

Let us finally address the limit $N \to \infty$. That this must be the general case of a quantum system is easily understood by the fact that there exist Observables with unbound and/or continuous spectra, and as in QFT the number of independent commuting Observables is infinite. Thus when passing from the general finite treatment I have just presented to the limit of Hilbert spaces proper we do not encounter new concepts, only new mathematical objects. Let us now see briefly what they are. The simplest case is that of a discrete unbound eigenvalue spectrum, which correspond to taking the infinite limit, that remains denumerable. In this case no new mathematical object arises, only the discrete indices $h, k \ldots$ span the infinite countable set of non-negative integers. Things are different when among the Observables of a *complete commuting set* there exist some that have a continuous eigenvalue spectrum. This happens, for instance, to the momentum operator of a quantum particle or field, defined in an unbound spatial region. The *trick* that the physicist does in this case, as a consequence of his *horror infiniti*, is to first *discretize* the continuum by introducing a *fundamental length* a, and keeping in the interval (o_0, o_n) only those points given by

$$o_k = ka + o_0 \qquad \left(k = 0, 1, \ldots, n; \frac{o_n - o_0}{n} = a\right) \tag{A.26}$$

In this way one recovers the discrete case and all the above developments. On the other hand the length a can be thought arbitrarily small, and the continuum limit is retrieved for $a \to 0$. Thus we must figure out how to represent in this limit operations such as $\frac{1}{n}\sum_k f_k$, or symbols like δ_{hk}; modern functional analysis (and Dirac himself) gives the answer. We have

clearly

$$\frac{1}{n}\sum_k f_k = \frac{1}{na}\sum_k af_k \to \frac{1}{o_n - o_0}\int_{o_0}^{o_n} do f(o)\,. \tag{A.27}$$

On the other hand for the Kronecker delta, from the identity:

$$\sum_k \delta_{hk} f_k = f_h \tag{A.28}$$

we have

$$\sum_k a\frac{\delta_{hk}}{a} f_k \to \int do\delta(o - o_h) f(o) \tag{A.29}$$

or

$$\frac{\delta_{hk}}{a} \to \delta(o - o')\,, \tag{A.30}$$

the celebrated Dirac delta function, a strange function (indeed a *distribution*) whose support is a single point, and being at that point infinite in such a way that the integral

$$\int do\delta(o - o') = 1\,. \tag{A.31}$$

But for our purposes it is time to stop here.

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